

Homogeneous Fedosov Star Products on Cotangent Bundles I: Weyl and Standard Ordering with Differential Operator Representation

Martin Bordemann^{*}, Nikolai Neumaier[§], Stefan Waldmann[¶]

Fakultät für Physik
Universität Freiburg
Hermann-Herder-Str. 3
79104 Freiburg i. Br., F. R. G

FR-THEP-97/10
July 1997

Abstract

In this paper we construct homogeneous star products of Weyl type on every cotangent bundle T^*Q by means of the Fedosov procedure using a symplectic torsion-free connection on T^*Q homogeneous of degree zero with respect to the Liouville vector field. By a fibrewise equivalence transformation we construct a homogeneous Fedosov star product of standard ordered type equivalent to the homogeneous Fedosov star product of Weyl type. Representations for both star product algebras by differential operators on functions on Q are constructed leading in the case of the standard ordered product to the usual standard ordering prescription for smooth complex-valued functions on T^*Q polynomial in the momenta (where an arbitrary fixed torsion-free connection ∇_0 on Q is used). Motivated by the flat case $T^*\mathbb{R}^n$ another homogeneous star product of Weyl type corresponding to the Weyl ordering prescription is constructed. The example of the cotangent bundle of an arbitrary Lie group is explicitly computed and the star product given by Gutt is rederived in our approach.

^{*}Martin.Bordemann@physik.uni-freiburg.de

[§]Nikolai.Neumaier@physik.uni-freiburg.de

[¶]Stefan.Waldmann@physik.uni-freiburg.de

1 Introduction

The concept of deformation quantization defined in [2] has now been well-established on every symplectic manifold: existence of the formal associative deformation of the pointwise multiplication of smooth functions, the so-called star product, had been shown by [11] and [15], and their classification up to equivalence transformations by formal power series in the second de Rham cohomology group is due to [18, 19] and [3].

The symplectic manifolds which are mostly used by physicists are cotangent bundles T^*Q of a smooth manifold Q , the configuration space of the classical dynamical system, which is rather often taken to be an open subset of \mathbb{R}^{2n} . There is a large amount of literature concerning star products on cotangent bundles (cf. e. g. [9], [10], [20]), differential operators and their symbolic calculus ([22],[23]), and also geometric quantization on cotangent bundles (see e. g. [25] and references therein).

The main motivation for us to write this paper was to apply the formal GNS construction in deformation quantization developed by two of us (cf. [6]) to the particular case of T^*Q : this method (which basically copies the standard GNS representation in the theory of C^* -algebras) allows to construct formal pre-Hilbert space representations of the associative algebra (over the field of formal Laurent series with complex coefficients, $\mathbb{C}((\lambda))$) of all formal Laurent series with coefficients in the space of all smooth complex-valued functions on a symplectic manifold equipped with a star product. The basic ingredient is a formally positive $\mathbb{C}((\lambda))$ -linear functional on this algebra. In case $Q = \mathbb{R}^n$ the usual Weyl ordered Schrödinger representation could thus be reconstructed by means of an integral over configuration space (see [6]) as well as the ordinary WKB expansion by means of a certain functional with support on a projectable Lagrangean submanifold of \mathbb{R}^{2n} contained in a classical energy surface (see [7]).

When starting to work on general cotangent bundles we realized that we had to develop first a good deal of compact practical formulas for certain star products on T^*Q and their possible representations as formal series with coefficients in differential operators on Q before we could start checking that even the simplest functional which consists in integration over Q (with respect to some volume in case Q is orientable) is formally positive. Therefore on one hand this paper will simply prepare the grounds for a second paper (see [4]) in which we shall define formal GNS representations on T^*Q and also compare our results with those obtained by analytic techniques (see e. g. [9], [20], [22], [23]). On the other hand we feel that some of our results, viz. a fairly explicit Fedosov construction on arbitrary T^*Q , and a rather simple closed formula relating a star product based on standard ordering and a particular star product of Weyl type (which is different from the ‘most natural’ Fedosov star product!) based on a generalization of Weyl ordering may well be of independent interest and useful in computations because the usual techniques of asymptotic expansions of certain integrals in normal coordinates are not needed.

Before summarizing our results let us first motivate our programme by the simple example $Q = \mathbb{R}^n$ in which everything can explicitly be computed (see e. g. [1], [6]):

The quantization method which is frequently used by physicists for $Q = \mathbb{R}^n$ in the Schrödinger picture proceeds as follows: Except for relativistic phase space functions such as the energy $\sqrt{m^2 + p^2}$ of a free particle in \mathbb{R}^n the great majority of classical observables occurring in physics are *polynomial in the momenta* p i. e. smooth functions $F : \mathbb{R}^{2n} \rightarrow \mathbb{C} : (q, p) \mapsto F(q, p)$ which take the form¹

$$(q, p) \mapsto F(q, p) = \sum_{k=0}^N \frac{1}{k!} F_k^{i_1 \dots i_k}(q) p_{i_1} \dots p_{i_k} \quad (1)$$

¹From now on we shall use the Einstein summation convention where the sum over repeated coordinate indices i_r where mostly $1 \leq i_r \leq n$ is automatic.

(where the $F_k^{i_1 \dots i_k}$ are smooth complex-valued functions on \mathbb{R}^n). On the space of all these functions which we shall denote by $C_{pp}^\infty(\mathbb{R}^{2n})$ (and which clearly forms a Poisson subalgebra of $C^\infty(\mathbb{R}^{2n})$) one establishes a linear bijection to the space of all differential operators on functions ψ on \mathbb{R}^n according to the following rule: a smooth complex-valued function $q \mapsto f(q)$ is mapped to the multiplication with f , i. e. $\psi \mapsto f\psi$, the coordinates p_l are mapped to $\frac{\hbar}{i} \frac{\partial}{\partial q^l}$ and for a general function polynomial in the momenta a so-called *ordering prescription* is applied to extend the map to a bijection: An important example is the *standard ordering prescription* where a function of the above form (1) is mapped to its *standard representation*, $\varrho_s(F)$, in the following way:

$$\varrho_s(F)(\psi) : q \mapsto \sum_{k=0}^N \frac{1}{k!} \left(\frac{\hbar}{i} \right)^k F_k^{i_1 \dots i_k}(q) \frac{\partial^k \psi}{\partial q^{i_1} \dots \partial q^{i_k}}(q). \quad (2)$$

The standard representation, however, is unphysical in the sense that the differential operators $\varrho_s(F)$ are not symmetric (when F takes real values) on, say, the space $\mathcal{D}(\mathbb{R}^n)$ of all smooth complex-valued functions with compact support with the standard inner product given by the Lebesgue integral (i. e. $\langle \phi, \psi \rangle := \int \overline{\phi(q)} \psi(q) d^n q$ where $\bar{}$ denotes pointwise complex conjugation): it is easily seen by induction and repeated partial integration that the formal adjoint $\varrho_s(F)^\dagger$ of $\varrho_s(F)$ (i. e. $\langle \varrho_s(F)^\dagger \phi, \psi \rangle = \langle \phi, \varrho_s(F) \psi \rangle$) is given by the differential operator

$$\varrho_s(F)^\dagger = \varrho_s(N^2 \overline{F}) \quad (3)$$

where

$$N := e^{\frac{\hbar}{2i} \frac{\partial^2}{\partial q^k \partial p_k}} \quad (4)$$

is a well-defined bijective linear map on all the functions polynomial in the momenta (1). These unphysical features can be remedied by defining the *Weyl representation* of F by

$$\varrho_w(F) := \varrho_s(NF) \quad (5)$$

which is also a bijection and clearly gives

$$\varrho_w(F)^\dagger = \varrho_w(\overline{F}) \quad (6)$$

such that real-valued functions are now mapped to symmetric operators. In case F is a polynomial function in p and in q it is not hard to see using the Baker-Campbell-Hausdorff series of the Heisenberg Lie algebra spanned by $1, \varrho_s(q^1), \dots, \varrho_s(q^n), \varrho_s(p_1), \dots, \varrho_s(p_n)$ that $\varrho_w(F)$ can be obtained by the so-called *Weyl ordering prescription* by means of which monomials are mapped to totally symmetrized operators (see e. g. [1], [6]), i. e.

$$q^{i_1} \dots q^{i_a} p_{j_1} \dots p_{j_b} \mapsto \frac{1}{(a+b)!} \sum_{\sigma \in S_{a+b}} A_{\sigma(1)} \dots A_{\sigma(a+b)}, \quad (7)$$

where $A_r := \varrho_s(q^{i_r})$ for all $1 \leq r \leq a$ and $A_r := \varrho_s(p_{j_{r-a}})$ for all $a+1 \leq r \leq a+b$.

The usual Moyal-Weyl star product $*_w$ in \mathbb{R}^{2n} of two smooth complex-valued functions F, G polynomial in the momenta,

$$(F *_w G)(q, p) := e^{\frac{i\hbar}{2} \left(\frac{\partial^2}{\partial q^i \partial p_i} - \frac{\partial^2}{\partial q^i \partial p_i} \right)} F(q, p) G(q', p') \Big|_{q=q', p=p'} \quad (8)$$

can for instance be obtained from the multiplication of the two Weyl representations, i. e.:

$$\varrho_w(F *_w G) = \varrho_w(F) \varrho_w(G). \quad (9)$$

Likewise, the multiplication of the two standard representations of F and G gives rise to another *star product of standard type*, $*_s$, in the following way (which makes sense since ϱ_s is a bijection into the space of all differential operators with smooth coefficients)

$$\varrho_s(F *_s G) := \varrho_s(F) \varrho_s(G) \quad (10)$$

and takes the following form:

$$F *_s G = \sum_{r=0}^{\infty} \frac{1}{r!} \left(\frac{\hbar}{i} \right)^r \frac{\partial^r F}{\partial p_{i_1} \dots \partial p_{i_r}} \frac{\partial^r G}{\partial q^{i_1} \dots \partial q^{i_r}}. \quad (11)$$

Due to (5) we clearly have equivalence of $*_s$ and $*_w$

$$F *_s G = N((N^{-1}F) *_w (N^{-1}G)). \quad (12)$$

Writing π for the canonical projection $\mathbb{R}^{2n} \rightarrow \mathbb{R}^n : (q, p) \mapsto q$ and i for the canonical zero section $\mathbb{R}^n \rightarrow \mathbb{R}^{2n} : q \mapsto (q, 0)$ we easily get the following useful formula for any smooth complex-valued function ψ on \mathbb{R}^n

$$\varrho_s(F)\psi = i^*(F *_s (\pi^*\psi)). \quad (13)$$

Considering \hbar now as an additional variable on which the functions depend polynomially we define the following differential operator

$$\mathcal{H} := p_i \frac{\partial}{\partial p_i} + \hbar \frac{\partial}{\partial \hbar}. \quad (14)$$

It is easy to see that both representations enjoy the following *homogeneity property*:

$$\left[\hbar \frac{\partial}{\partial \hbar}, \varrho_{s/w}(F) \right] = \varrho_{s/w}(\mathcal{H}F). \quad (15)$$

Physically this means that the operator corresponding to the momentum component p_l has also the physical dimension of a momentum which is equal to the dimension of \hbar divided by length (the dimension of q^l). As a consequence, the two star products are also homogeneous in the sense that the map \mathcal{H} is a derivation:

$$\mathcal{H}(F *_s G) = (\mathcal{H}F) *_s G + F *_s (\mathcal{H}G). \quad (16)$$

Both star products are associative and deform the pointwise multiplication such that the component of the commutator which is first order in \hbar equals i times the canonical Poisson bracket. Moreover, as can easily be seen by the formulas (8) and (11) both star products are *bidifferential* in each order of \hbar . They are even of *Vey type* which means that in order \hbar^r the corresponding bidifferential operator is of order r in each argument. One might be tempted to think that any ordering prescription may give rise to a reasonable star product, but the following example indicates that one may lose the property that the star product be bidifferential in each order of \hbar :

For \mathbb{R}^2 define the following modified ordering prescription for a positive real number s :

$$\varrho_{\text{perv}}((q, p) \mapsto f(q)p^k) := \begin{cases} \varrho_s((q, p) \mapsto f(q)p^k) & \text{if } k \neq 2 \\ \varrho_s((q, p) \mapsto f(q)p^2) + s \frac{\hbar}{i} \varrho_s((q, p) \mapsto f(q)p) & \text{if } k = 2 \end{cases} \quad (17)$$

A lengthy, but straight forward computation using formal symbols $F(q, p) := e^{\alpha q + \beta p}$ where β is considered as a formal parameter shows that in the corresponding star product (which is well-defined on all smooth functions polynomial in the momenta), i. e. $\varrho_{\text{perv}}(F) \varrho_{\text{perv}}(G) = \varrho_{\text{perv}}(F *_s G)$,

for each order in \hbar there are infinitely many derivatives with respect to p . Hence this kind of star product is in general *not* extendable to the space of all formal power series in \hbar with coefficients in the smooth complex-valued functions on \mathbb{R}^2 .

The aim is now to construct and compute in more detail concrete star products on arbitrary cotangent bundles T^*Q and possible representations as formal series of differential operators on the formal series of smooth complex-valued functions on Q . The important feature of these structures will be their *homogeneity in the momenta* which is now defined using the Liouville vector field ξ on T^*Q (whose flow consists in multiplying the fibres by e^t) and which takes the familiar form $p_i \frac{\partial}{\partial p_i}$ in a bundle chart. It is interesting to note that the existence proof for star products on arbitrary cotangent bundles T^*Q by DeWilde and Lecomte [10] in 1983 is much easier than the general proof thanks to this notion of homogeneity (which had earlier been used by Cahen and Gutt for parallelizable manifolds, see [8]): By demanding the bidifferential operators M_r in the formal series of the star product of two smooth complex-valued functions f and g ,

$$f * g = \sum_{r=0}^{\infty} \hbar^r M_r(f, g), \quad (18)$$

to be homogeneous of order $-r$ with respect to the Liouville field DeWilde and Lecomte were able to show that the usual obstructions in the third de Rham cohomology which a priori occur when constructing the M_r by induction (see e. g. [12]) simply vanish due to the homogeneity requirement.

The immediate generalization of the standard representation ϱ_s to an arbitrary cotangent bundle T^*Q proceeds as follows: take the space of sections $\Gamma(\bigvee^k TQ)$ of the k -fold symmetrized tangent bundle and consider the canonical linear injection $\hat{\cdot} : \Gamma(\bigvee^k TQ) \rightarrow C^\infty(T^*Q) : T \mapsto \hat{T} : \alpha \mapsto \frac{1}{k!} T(\alpha, \dots, \alpha)$. We call the complexification of its image $C_{pp,k}^\infty(T^*Q)$, and the direct sum $\bigoplus_{k=0}^\infty C_{pp,k}^\infty(T^*Q)$ is denoted by $C_{pp}^\infty(T^*Q)$ which is the obvious analogue of the smooth complex-valued functions polynomial in the momenta. Clearly $\mathcal{L}_\xi F = kF$ iff F in $C_{pp,k}^\infty(T^*Q)$. Moreover choose a torsion-free covariant derivative ∇_0 in the tangent bundle of Q and replace partial by covariant derivatives in (2) in the following manner: for $T = T_1 + \dots + T_N$ where $T_k \in \Gamma(\bigvee^k TQ)$ ($0 \leq k \leq N$) we define for a smooth complex-valued function ψ on Q

$$\varrho_s(\hat{T})\psi : q \mapsto \sum_{k=0}^N \frac{1}{k!} \left(\frac{\hbar}{i} \right)^k T_k^{i_1 \dots i_k}(q) i_s(\partial_{q^{i_1}}) \dots i_s(\partial_{q^{i_k}}) D_0^{(k)} \psi(q) \quad (19)$$

where $D_0^{(k)}$ is the k -fold symmetrized covariant derivative, i_s means symmetric substitution and the functions $T_k^{i_1 \dots i_k}$ are regarded as the components of the contravariant symmetric tensor field T_k on Q . This will clearly induce a star product on the commutative subalgebra $C_{pp}^\infty(T^*Q)$ of $C^\infty(T^*Q)$. However, since the Christoffel symbols of the connection ∇_0 modify the differential operators of flat space representation by terms of lower order it is a priori questionable in view of the above counterexample whether this star product is bidifferential at each order of \hbar . In fact it turns out that it is bidifferential (which seems to have been shown by hard analytic techniques in the past, cf. e. g. [20]) and in this paper we want to deal with this question in a more algebraic manner. Our main results are the following:

- After giving a short generalizing review of the Fedosov construction on an arbitrary manifold in Section 2 (where we need the covariant derivative term in more generality and a fibrewise formulation of equivalence transformation) we then build up and compute the general Fedosov machinery for a star product of standard ordered $*_s$ and Weyl type $*_F$ on any cotangent bundle T^*Q (Section 3 and 4) based on a rather natural, seemingly well-known lift of any

torsion-free covariant derivative on Q to a symplectic torsion-free covariant derivative on T^*Q which is homogeneous with respect to the Liouville vector field on T^*Q (see Appendix A for a description). It turns out that the corrections to the covariant derivative needed to construct the Fedosov derivative are “classical” in the sense that they do *not* depend on the formal parameter. The standard ordered type fibrewise star product is constructed by using the duality between the horizontal and the vertical subbundle of the tangent bundle of T^*Q . Then we can prove by constructing an equivalence transformation in Section 5 that the Fedosov star product of standard ordered type \ast_s is equivalent to the Fedosov star product of Weyl type \ast_F .

- According to the general Fedosov philosophy “Whatever you plan to do on a symplectic manifold M , do it first fibrewise on the tangent spaces, and pull it then down to M by means of a nice compatible Fedosov derivative” we are then constructing a fibrewise standard representation analogous to the flat space formula (13) in Section 6. As a surprise it turned out that our naively constructed Fedosov derivative \mathcal{D}' (66) (as a conjugate of the original Weyl type Fedosov derivative \mathcal{D}_F by means of the fibrewise analogue to the operator N , see (4)) was *not* compatible with the fibrewise standard representation. Luckily, it could be modified by a fibrewise internal automorphism (see Thm. 5.2) to render it compatible. As a result we obtain exactly the above standard representation (19). Since it is easy to see that all ‘reasonable’ star products constructed by a Fedosov type procedure automatically are bidifferential the construction shows a posteriori that the standard representation does not fall under the above-mentioned “beasty” class of ordering prescriptions.
- Finally we derive a surprisingly simple analogue of the operator N (cf. (4)) for any T^*Q in Section 7: it takes the form $N = \exp(\frac{\hbar}{2i}\Delta)$ where the second-order differential operator Δ takes the following form in a bundle chart (q, p) :

$$(\Delta F)(q, p) = \frac{\partial^2 F}{\partial q^i \partial p_i}(q, p) + \Gamma_{ik}^i(q) \frac{\partial F}{\partial p_k}(q, p) + p_r \Gamma_{ij}^r(q) \frac{\partial^2 F}{\partial p_i \partial p_j}(q, p) + \alpha_r(q) \frac{\partial F}{\partial p_r}(q, p) \quad (20)$$

where the Γ_{jk}^i are the Christoffel symbols of the connection ∇_0 and α is a particular choice of a one-form on Q such that $-d\alpha$ equals the trace of the curvature tensor (see the Appendix for a Theorem). In case ∇_0 leaves invariant a volume on Q (assumed to be orientable) then α can be chosen to be zero. N can now be used as an equivalence transformation from the star product of standard ordered type \ast_s to another star product \ast_w (107) which is of Weyl type but which turns out to be *different* from the Fedosov star product of Weyl type, \ast_F !

As an example, we rederive the star product on the cotangent bundle of an arbitrary Lie group constructed by means of the standard torsion-free left-invariant ‘half commutator’ connection first given by Gutt (see [17]) and give an explicit closed formula of a star product of standard ordered type in Section 8.

Convention: In what follows \hbar will always denote a real number whereas λ will denote a formal parameter which in converging situations may be substituted by \hbar and is considered to be real, i. e. $\overline{\lambda} := \lambda$.

2 Fedosov Derivations and Fedosov-Taylor series

In this rather technical section we shall revisit Fedosov’s construction of star products in a slightly more general context. The notation is mainly the same as in Fedosov’s book [16] and in [5]. Let

M be a smooth manifold and define

$$\mathcal{W} \otimes \Lambda(M) := (\mathbf{X}_{s=0}^\infty \mathbb{C}(\Gamma(\bigvee^s T^*M \otimes \bigwedge T^*M))) [[\lambda]]. \quad (21)$$

If there is no possibility for confusion we simply write $\mathcal{W} \otimes \Lambda$ and denote by $\mathcal{W} \otimes \Lambda^k$ the elements of antisymmetric degree k and set $\mathcal{W} := \mathcal{W} \otimes \Lambda^0$. For two elements $a, b \in \mathcal{W} \otimes \Lambda$ we define their pointwise product denoted by $\mu(a \otimes b) = ab$ by the symmetric \vee -product in the first factor and the antisymmetric \wedge -product in the second factor. Then the degree-maps \deg_s and \deg_a with respect to the symmetric and antisymmetric degree are derivations of this product. Therefore we shall call $\mathcal{W} \otimes \Lambda$ a formally $\mathbb{Z} \times \mathbb{Z}$ -graded algebra with respect to the symmetric and antisymmetric degree. Moreover $(\mathcal{W} \otimes \Lambda, \mu)$ is supercommutative with respect to the antisymmetric degree. For a vector field X we define the symmetric substitution (insertion) $i_s(X)$ and the antisymmetric substitution $i_a(X)$ which are superderivations of symmetric degree -1 resp. 0 and antisymmetric degree 0 resp. -1 . Following Fedosov we define

$$\delta := (1 \otimes dx^i) i_s(\partial_{x^i}) \quad \text{and} \quad \delta^* := (dx^i \otimes 1) i_a(\partial_{x^i}) \quad (22)$$

where x^1, \dots, x^n are local coordinates for M and for $a \in \mathcal{W} \otimes \Lambda$ with $\deg_s a = ka$ and $\deg_a a = la$ we define

$$\delta^{-1} a := \begin{cases} \frac{1}{k+l} \delta^* a & \text{if } k+l \neq 0 \\ 0 & \text{if } k+l = 0 \end{cases} \quad (23)$$

and extend δ^{-1} by linearity. Clearly $\delta^2 = \delta^{*2} = 0$. Moreover we denote by $\sigma : \mathcal{W} \otimes \Lambda \rightarrow C^\infty(M) [[\lambda]]$ the projection onto the part of symmetric and antisymmetric degree 0 . Then one has the following ‘Hodge-decomposition’ for any $a \in \mathcal{W} \otimes \Lambda$ (see e. g. [15, eq. 2.8.]):

$$a = \delta \delta^{-1} a + \delta^{-1} \delta a + \sigma(a) \quad (24)$$

Now we consider a fibrewise associative deformation \circ of the pointwise product which should have the form

$$a \circ b = ab + \sum_{r=1}^{\infty} \lambda^r \mathcal{M}_r(a, b) \quad (25)$$

where $\mathcal{M}_r(a, b) = M_r^{i_1 \dots i_r j_1 \dots j_r} i_s(\partial_{x^{i_1}}) \dots i_s(\partial_{x^{i_r}}) a i_s(\partial_{x^{j_1}}) \dots i_s(\partial_{x^{j_r}}) b$ and the $M_r^{i_1 \dots i_r j_1 \dots j_r}$ are the coefficients of a tensor field totally symmetric in i_1, \dots, i_r and j_1, \dots, j_r separately. Moreover we define \deg_a -graded supercommutators with respect to \circ and set $\text{ad}(a)b := [a, b]$. If not all $\mathcal{M}_r = 0$ for $r \geq 1$ then \deg_s is no longer a derivation of the deformed product \circ but $\text{Deg} := \deg_s + 2\deg_\lambda$ is still a derivation and hence the algebra $(\mathcal{W} \otimes \Lambda, \circ)$ is formally Deg -graded where $\deg_\lambda := \lambda \partial_\lambda$. We shall refer to this degree as total degree. We shall treat the non-deformed case separately at the end of this section and first remember the following two theorems which can be proved completely analogously to Fedosov’s original theorems in [15, Theorem 3.2, 3.3]:

Theorem 2.1 *Let $T^{(0)} : \mathcal{W} \otimes \Lambda \rightarrow \mathcal{W} \otimes \Lambda$ be a superderivation of \circ of antisymmetric degree 1 and total degree 0 such that $[\delta, T^{(0)}] = 0$, and $(T^{(0)})^2 = \frac{1}{2}[T^{(0)}, T^{(0)}] = \frac{i}{\lambda} \text{ad}(T)$ with some $T \in \mathcal{W} \otimes \Lambda^2$ of total degree 2, and let T satisfy $\delta T = 0 = T^{(0)} T$. Then there exists a unique element $r \in \mathcal{W} \otimes \Lambda^1$ such that*

$$\delta r = T + T^{(0)} r + \frac{i}{\lambda} r \circ r \quad \text{and} \quad \delta^{-1} r = 0. \quad (26)$$

Moreover $r = \sum_{k=3}^{\infty} r^{(k)}$ with $\text{Deg} r^{(k)} = k r^{(k)}$ satisfies the recursion formulas

$$\begin{aligned} r^{(3)} &= \delta^{-1} T \\ r^{(k+3)} &= \delta^{-1} \left(T^{(0)} r^{(k+2)} + \frac{i}{\lambda} \sum_{l=1}^{k-1} r^{(l+2)} \circ r^{(k-l+2)} \right). \end{aligned} \quad (27)$$

In this case the Fedosov derivation

$$\mathcal{D} := -\delta + T^{(0)} + \frac{i}{\lambda} \text{ad}(r) \quad (28)$$

is a superderivation of antisymmetric degree 1 and has square zero: $\mathcal{D}^2 = 0$.

Theorem 2.2 Let $L = -\delta + T : \mathcal{W} \otimes \Lambda \rightarrow \mathcal{W} \otimes \Lambda$ be a $\mathbb{C}[[\lambda]]$ -linear map of antisymmetric degree 1 with square zero $L^2 = 0$ such that T does not decrease the total degree.

i.) Then for any $f \in C^\infty(M)[[\lambda]]$ there exists a unique element $\tau_L(f) \in \ker L \cap \mathcal{W}$ such that

$$\sigma(\tau_L(f)) = f \quad (29)$$

and $\tau_L : C^\infty(M)[[\lambda]] \rightarrow \mathcal{W}$ is $\mathbb{C}[[\lambda]]$ -linear and referred to as the Fedosov-Taylor series corresponding to L .

ii.) If in addition $T = \sum_{k=0}^{\infty} T^{(k)}$ such that $T^{(k)}$ is homogeneous of total degree k then for $f \in C^\infty(M)$ we have $\tau_L(f) = \sum_{k=0}^{\infty} \tau_L(f)^{(k)}$ where $\text{Deg} \tau_L(f)^{(k)} = k \tau_L(f)^{(k)}$ which can be obtained by the following recursion formula

$$\begin{aligned} \tau_L(f)^{(0)} &= f \\ \tau_L(f)^{(k+1)} &= \delta^{-1} \sum_{l=0}^k T^{(l)} \tau_L(f)^{(k-l)}. \end{aligned} \quad (30)$$

iii.) If $L = \mathcal{D}$ is a \circ -superderivation of antisymmetric degree 1 as constructed in theorem 2.1 then $\ker \mathcal{D} \cap \mathcal{W}$ is a \circ -subalgebra and a new (eventually deformed) associative product $*_{\mathcal{D}}$ for $C^\infty(M)[[\lambda]]$ is defined by pull-back of \circ via $\tau_{\mathcal{D}}$.

Let \circ' be another fibrewise product for $\mathcal{W} \otimes \Lambda$ and $\mathcal{S} = \text{id} + \sum_{r=1}^{\infty} \lambda^r \mathcal{S}_r$ where $\mathcal{S}_r : \mathcal{W} \otimes \Lambda \rightarrow \mathcal{W} \otimes \Lambda$ is a map of the form $\mathcal{S}_r = \sum_{l=1}^{2r} \frac{1}{l!} S^{i_1 \dots i_l} i_s(\partial_{x^{i_1}}) \dots i_s(\partial_{x^{i_l}})$ where $S^{i_1 \dots i_l}$ are the components of a symmetric tensor field. If in addition

$$\mathcal{S}(a \circ b) = (\mathcal{S}a) \circ' (\mathcal{S}b)$$

for all $a, b \in \mathcal{W} \otimes \Lambda$ then \mathcal{S} is called a fibrewise equivalence transformation between \circ and \circ' . Note that \mathcal{S} is clearly invertible and \mathcal{S}^{-1} is a fibrewise equivalence transformation between \circ' and \circ . If \mathcal{D} is a \circ -superderivation as constructed in theorem 2.1 and $\tau_{\mathcal{D}}$ its corresponding Fedosov-Taylor series and $*_{\mathcal{D}}$ the induced associative product for $C^\infty(M)[[\lambda]]$ then $\mathcal{D}' := \mathcal{S} \mathcal{D} \mathcal{S}^{-1}$ is a \circ' -superderivation of antisymmetric degree 1 of the form $\mathcal{D}' = -\delta + T'$ satisfying the conditions of part one of the preceding theorem. Hence there exists a corresponding Fedosov-Taylor series $\tau_{\mathcal{D}'}$ which induces an associative product $*_{\mathcal{D}'}$ on $C^\infty(M)[[\lambda]]$. Then $*_{\mathcal{D}}$ and $*_{\mathcal{D}'}$ turn out to be equivalent too:

Proposition 2.3 With the notation from above we define the map $T : C^\infty(M)[[\lambda]] \rightarrow C^\infty(M)[[\lambda]]$

$$Tf := \sigma(\mathcal{S} \tau_{\mathcal{D}}(f)) \quad (31)$$

for $f \in C^\infty(M)[[\lambda]]$ which is an $\mathbb{C}[[\lambda]]$ -linear equivalence transformation between $*_{\mathcal{D}}$ and $*_{\mathcal{D}'}$, i. e. $T(f *_{\mathcal{D}} g) = (Tf) *_{\mathcal{D}'} (Tg)$ for all $f, g \in C^\infty(M)[[\lambda]]$ with inverse $T^{-1}f = \sigma(\mathcal{S}^{-1} \tau_{\mathcal{D}'}(f))$.

PROOF: This is a straight forward computation observing $\mathcal{D}'\mathcal{S}\tau_{\mathcal{D}}(f) = 0$ and applying the last theorem. \square

At last we shall discuss the classical case with the undeformed product μ . First we restrict our considerations to the classical part $\mathcal{W} \otimes \Lambda_{\text{cl}}$ of $\mathcal{W} \otimes \Lambda$ which are just those elements without any positive λ -powers. Next we consider a torsion-free connection ∇ for M and define the map (using the same symbol as for the connection)

$$\nabla := (1 \otimes dx^i) \nabla_{\partial_{x^i}} \quad (32)$$

where $\nabla_{\partial_{x^i}}$ denotes the covariant derivative with respect to ∂_{x^i} . Then clearly ∇ is globally defined and a superderivation of μ which leaves $\mathcal{W} \otimes \Lambda_{\text{cl}}$ invariant. Moreover we consider

$$\mathcal{W} \otimes \Lambda \otimes \mathcal{X}(M) := (\mathbf{X}_{s=0}^\infty \mathbb{C}(\Gamma(\bigvee^s T^*M \otimes \bigwedge T^*M \otimes TM))) [[\lambda]] \quad (33)$$

and define $\mathcal{W} \otimes \Lambda \otimes \mathcal{X}_{\text{cl}}$ analogously. For $\varrho \in \mathcal{W} \otimes \Lambda \otimes \mathcal{X}$ we define the symmetric substitution $i_s(\varrho)$ by inserting the vector part of ϱ symmetrically and multiplying the form part of ϱ by μ from the left. Then we have $\nabla^2 = -i_s(R)$ where R is the curvature tensor viewed as element of antisymmetric degree 2 in $\mathcal{W} \otimes \Lambda \otimes \mathcal{X}$. The classical analogue to the Fedosov derivation is described by the following theorem which is due to Emmerich and Weinstein [14, Theorem 1]:

Theorem 2.4 *Let ∇ be defined as in (32) then ∇ is a superderivation of the undeformed product μ and there exists a uniquely determined element $\varrho_0 \in \mathcal{W} \otimes \Lambda \otimes \mathcal{X}_{\text{cl}}$ of antisymmetric degree 1 such that $\delta^{-1}\varrho_0 = 0$ and such that the classical Fedosov derivation*

$$\mathcal{D}_0 = -\delta + \nabla + i_s(\varrho_0) \quad (34)$$

has square zero: $\mathcal{D}_0^2 = 0$. Note that $\mathcal{D}_0\varrho_0 = R_0$ and $\overline{\varrho_0} = \varrho_0$ as well as $\overline{\mathcal{D}_0 a} = \mathcal{D}_0 \bar{a}$ for all $a \in \mathcal{W} \otimes \Lambda$.

Furthermore in [14, Theorem 3 and 6] was shown using analytical techniques that in this case the corresponding Fedosov-Taylor series τ_0 is just the formal Taylor series with respect to the connection. We shall give here another more algebraic proof of this result:

Theorem 2.5 *Let τ_0 be the Fedosov-Taylor series of \mathcal{D}_0 according to theorem 2.2. Then for $f \in C^\infty(M)$ we have*

$$\tau_0(f) = e^D f \quad (35)$$

where $D = dx^i \vee \nabla_{\partial_{x^i}}$ and hence $\tau_0(f)$ is the formal Taylor series with respect to the connection ∇ .

PROOF: Since \mathcal{D}_0 satisfies all conditions for theorem 2.2 we only have to show that $e^D f$ satisfies the recursion formula (30). Note that in this particular case the total degree and the symmetric degree coincide. First we observe $D = [\delta^*, \nabla]$ and hence applying δ and δ^* to (30) implies $0 = -(\delta^* \delta + \delta \delta^*) \tau_0(f)^{(k+1)} + D \tau_0(f)^{(k)} + \sum_{l=0}^k \delta^* \left(i_s(\varrho_0^{(l+1)}) \tau_0(f)^{(k-l)} \right)$. Now each term in the last sum vanishes identically due to $\delta^* \varrho_0 = 0$ and $\deg_a \tau_0(f) = 0$ hence $(k+1) \tau_0(f)^{(k+1)} = D \tau_0(f)^{(k)}$ since $\delta^* \delta + \delta \delta^* = \deg_s + \deg_a$. Then (35) follows directly by induction on the symmetric degree k . \square

3 Homogeneous Fedosov star product of Weyl type

Let $\pi : T^*Q \rightarrow Q$ be the cotangent bundle of a differentiable, n -dimensional manifold Q and let θ_0 be the canonical one-form, $\omega_0 := -d\theta_0$ the canonical symplectic form and ξ defined by $i_\xi \omega_0 = -\theta_0$ the canonical (Liouville) vector field on T^*Q . Moreover let $i : Q \rightarrow T^*Q$ be the embedding of Q in T^*Q as zero section. We consider now for $\mathcal{W} \otimes \Lambda$ of T^*Q the fibrewise Weyl product defined for $a, b \in \mathcal{W} \otimes \Lambda$ by

$$a \circ_F b := \mu \circ e^{\frac{i\lambda}{2} \Lambda^{kl} i_s(\partial_{x^k}) \otimes i_s(\partial_{x^l})} a \otimes b \quad (36)$$

where $\mu(a \otimes b) = ab$ is the fibrewise product in $\mathcal{W} \otimes \Lambda$ and Λ^{kl} are the components of the canonical Poisson tensor with respect to some coordinates x^1, \dots, x^{2n} of T^*Q . It will be advantageous for calculations to use local bundle (Darboux) coordinates $q^1, \dots, q^n, p_1, \dots, p_n$ induced by coordinates q^1, \dots, q^n on Q such that p_1, \dots, p_n are the conjugate momenta to the q^1, \dots, q^n . In the following $q^1, \dots, q^n, p_1, \dots, p_n$ shall always denote such a local bundle (Darboux) chart. Obviously \circ_F is an associative deformation of μ of the form as in (25) and hence we can apply all results of section 2 to this particular situation. We denote by \mathcal{L}_ξ the Lie derivative with respect to the canonical vector field ξ and define the ‘homogeneity derivation’

$$\mathcal{H} := \mathcal{L}_\xi + \deg_\lambda = \mathcal{L}_\xi + \lambda \frac{\partial}{\partial \lambda}. \quad (37)$$

Note that \mathcal{H} is \mathbb{C} -linear but not $\mathbb{C}[[\lambda]]$ -linear. Then the fibrewise Weyl product is homogeneous which can be proved as in the case of \mathbb{R}^{2n} :

Lemma 3.1 *Let $a, b \in \mathcal{W} \otimes \Lambda$ then \mathcal{H} is a (super-)derivation of \circ_F of antisymmetric and total degree 0:*

$$\mathcal{H}(a \circ_F b) = \mathcal{H}a \circ_F b + a \circ_F \mathcal{H}b \quad (38)$$

Moreover we have $[\mathcal{L}_\xi, \delta] = [\mathcal{L}_\xi, \delta^*] = [\mathcal{L}_\xi, \delta^{-1}] = 0$ and $[\mathcal{H}, \delta] = [\mathcal{H}, \delta^*] = [\mathcal{H}, \delta^{-1}] = 0$.

According to Fedosov’s construction of a star product one needs a torsion-free and symplectic connection ∇ for T^*Q and the map $\nabla : \mathcal{W} \otimes \Lambda \rightarrow \mathcal{W} \otimes \Lambda$ defined as in (32). If the connection is symplectic ∇ turns out to be a superderivation of antisymmetric degree 1 and symmetric and total degree 0 of the fibrewise Weyl product \circ_F . Moreover $[\delta, \nabla] = 0$ and $2\nabla^2 = [\nabla, \nabla]$ turns out to be an inner superderivation

$$\nabla^2 = \frac{i}{\lambda} \text{ad}_F(R) \quad (39)$$

where $R := \frac{1}{4} \omega_{it} R_{jkl}^t dx^i \vee dx^j \otimes dx^k \wedge dx^l \in \mathcal{W} \otimes \Lambda^2$ involves the curvature of the connection. Moreover one has $\delta R = 0 = \nabla R$ as a consequence of the Bianchi identities. Hence the map ∇ as term of total degree 0 satisfies all conditions of theorem 2.1 and hence there is a unique element $r_F \in \mathcal{W} \otimes \Lambda^1$ such that $\delta r_F = R + \nabla r_F + \frac{i}{\lambda} r_F \circ_F r_F$ and $\delta^{-1} r_F = 0$. Moreover the Fedosov derivation

$$\mathcal{D}_F := -\delta + \nabla + \frac{i}{\lambda} \text{ad}_F(r_F) \quad (40)$$

has square 0. Let τ_F be the corresponding Fedosov-Taylor series. Then Fedosov has shown that

$$f *_F g := \sigma(\tau_F(f) \circ_F \tau_F(g)) \quad (41)$$

defines a star product [15, eq. 3.14] which is of Weyl type (see e. g. [5, Lemma 3.3]).

In the particular case of a cotangent bundle we consider homogeneous, symplectic and torsion-free connections (see definition A.3):

Lemma 3.2 *If the symplectic and torsion-free connection ∇ on T^*Q is in addition homogeneous then*

$$[\mathcal{L}_\xi, \nabla] = [\mathcal{H}, \nabla] = 0 \quad (42)$$

$$\mathcal{H}R = \mathcal{L}_\xi R = R. \quad (43)$$

Theorem 3.3 *Let ∇ be defined as above with the additional property that the connection is homogeneous. Then*

$$\mathcal{L}_\xi r_F = r_F = \mathcal{H}r_F \quad \text{and} \quad \frac{\partial}{\partial \lambda} r_F = 0. \quad (44)$$

Moreover the Fedosov derivation \mathcal{D}_F (super-)commutes with \mathcal{H}

$$[\mathcal{H}, \mathcal{D}_F] = 0 \quad (45)$$

and r_F satisfies the following simpler recursion formulas

$$\begin{aligned} r_F^{(3)} &= \delta^{-1} R \\ r_F^{(k+3)} &= \delta^{-1} \left(\nabla r_F^{(k+2)} - \frac{1}{2} \sum_{l=1}^{k-1} \left\{ r_F^{(l+2)}, r_F^{(k-l+2)} \right\}_{\text{fib}} \right) \end{aligned} \quad (46)$$

where $\{\cdot, \cdot\}_{\text{fib}}$ denotes the fibrewise Poisson bracket in $\mathcal{W} \otimes \Lambda$. Moreover the corresponding Fedosov-Taylor series τ_F commutes with \mathcal{H}

$$\mathcal{H}\tau_F(f) = \tau_F(\mathcal{H}f) \quad (47)$$

*and satisfies the usual recursion formulas for $f \in C^\infty(T^*Q)$ with respect to the total degree*

$$\begin{aligned} \tau_F(f)^{(0)} &= f \\ \tau_F(f)^{(k+1)} &= \delta^{-1} \left(\nabla \tau_F(f)^{(k)} + \frac{i}{\lambda} \sum_{l=1}^{k-1} \text{ad}_F \left(r_F^{(l+2)} \right) \tau_F(f)^{(k-l)} \right) \end{aligned} \quad (48)$$

*analogously to (30). The Fedosov star product (41) is homogeneous, i. e. for $f, g \in C^\infty(T^*Q)[[\lambda]]$*

$$\mathcal{H}(f *_F g) = (\mathcal{H}f) *_F g + f *_F (\mathcal{H}g). \quad (49)$$

PROOF: The fact $\mathcal{H}r_F = r_F$ is proved by induction using the recursion formulas for r_F . Then (45), (47) and (49) easily follow and (48) follows directly from (30). Since $\mathcal{H}r_F = r_F$ the section r_F can depend at most linearly on λ but since it has to depend on even powers of λ only (see [5, Lemma 3.3]) it has to be independent of λ at all. Then the recursion formulas (46) follow by induction. \square

We shall refer to $*_F$ as the *homogeneous Fedosov star product of Weyl type* induced by the homogeneous connection ∇ . Using this theorem we find several corollaries. The first one is originally due to DeWilde and Lecomte [10, Proposition 4.1]:

Corollary 3.4 *On every cotangent bundle T^*Q there exists a homogeneous star product of Weyl type.*

Corollary 3.5 *Let $f \in C_{pp,k}^\infty(T^*Q)$ then $\tau_F(f)$ contains only even powers of λ up to order k .*

PROOF: This easily follows from (47) and [5, Lemma 3.3]. \square

Corollary 3.6 *The Fedosov-Taylor series τ_F satisfies*

$$\tau_F \circ \pi^* = \pi^* \circ i^* \circ \tau_F \circ \pi^*. \quad (50)$$

General properties of homogeneous star products are described in the following proposition:

Proposition 3.7 *Let $*$ be a homogeneous star product for T^*Q .*

- i.) The functions polynomial in λ and in the momenta $C_{pp}^\infty(T^*Q)[\lambda]$ are a $\mathbb{C}[\lambda]$ -submodule of $C^\infty(T^*Q)[[\lambda]]$ with respect to $*$ and hence $f * g$ trivially converges for all $\lambda = \hbar \in \mathbb{R}$ and $f, g \in C_{pp}^\infty(T^*Q)[\lambda]$.
- ii.) Let $U \subseteq Q$ be a domain of a chart. Then any $f \in C_{pp}^\infty(T^*U)[\lambda]$ can be written as finite sum of star products of functions in $C_{pp,0}^\infty(T^*U)[\lambda]$ and $C_{pp,1}^\infty(T^*U)[\lambda]$.
- iii.) The vector space $C_p^\omega(T^*Q)[[\lambda]]$ of formal power series with coefficients in the functions which are analytic in the fibre variables are a $\mathbb{C}[[\lambda]]$ -submodule of $C^\infty(T^*Q)[[\lambda]]$.

PROOF: The first part is obvious using lemma A.2 and for the third part one observes that the homogeneity implies that the coefficient functions of the bidifferential operators in the star product are polynomial in the momenta. For the second part note that $f \in C_{pp,k+1}^\infty(T^*U)$ takes the form $f(q, p) = \frac{1}{(k+1)!} F^{i_1 \dots i_{k+1}}(q) p_{i_1} \dots p_{i_{k+1}}$. Subtracting $\frac{1}{(k+1)!} (\pi^* F^{i_1 \dots i_{k+1}}) * p_{i_1} * \dots * p_{i_{k+1}}$ results in a finite sum of functions of degree smaller than $k+1$ with respect to \mathcal{L}_ξ and polynomial in λ proving the obvious induction on k . \square

Now we consider the Fedosov derivation \mathcal{D}_F and the section r_F more closely. First we notice that δ , δ^* and δ^{-1} satisfy the following relations

$$\delta\pi^* = \pi^*\delta_0 \quad \delta^*\pi^* = \pi^*\delta_0^* \quad \delta^{-1}\pi^* = \pi^*\delta_0^{-1} \quad (51)$$

where δ_0 , δ_0^* and δ_0^{-1} are the corresponding maps on Q defined analogously to δ , δ^* and δ^{-1} . Moreover for a homogeneous connection we get

$$\nabla\pi^* = \pi^*\nabla_0 \quad (52)$$

where ∇_0 is the corresponding map on Q defined by the induced connection ∇_0 on Q (see definition A.5). By direct calculation we get the following proposition:

Proposition 3.8 *Let \mathcal{D}_F and r_F be given as in (40). Then there exists a unique element $\varrho \in \mathcal{W} \otimes \Lambda \otimes \mathcal{X}_{\text{cl}}(Q)$ of antisymmetric degree 1 such that*

$$\mathcal{D}_F\pi^* = \pi^*\mathcal{D} \quad (53)$$

where $\mathcal{D} = -\delta_0 + \nabla_0 + i_s(\varrho)$ and clearly $\mathcal{D}^2 = 0$. In local coordinates the element ϱ takes the following form

$$\varrho = i^*(i_s(\partial_{p_i})r_F) \otimes \partial_{q^i} \quad (54)$$

and we have $\delta_0^*\varrho = 0$ iff $i_a(X)r_F = 0$ for all vertical vector fields $X \in \Gamma(T(T^*Q))$.

With other words, if $i_a(X)r_F = 0$ for all vertical vector fields then \mathcal{D} would coincide with the map \mathcal{D}_0 and ϱ would coincide with ϱ_0 as in theorem 2.4 applied for $M = Q$. In the following we shall prove that for any torsion-free connection on Q there is indeed a canonical choice for a homogeneous connection on T^*Q such that this is the case:

Proposition 3.9 *Consider a torsion-free connection on Q and the map ∇_0 defined as in (32) and let ∇ be a homogeneous, symplectic and torsion-free connection for T^*Q such that $\nabla\pi^* = \pi^*\nabla_0$ and let r_F be the corresponding element in $\mathcal{W} \otimes \Lambda^1$. Then $i_a(X)r_F^{(3)} = 0$ for every vertical vector field $X \in \Gamma(T(T^*Q))$ where $r_F^{(3)}$ is the term of total degree 3 in r_F iff the connection ∇ coincides with the lifted connection ∇^0 defined as in (A.4). Moreover in this case*

$$i_a(X)r_F = 0. \quad (55)$$

PROOF: The fact $i_a(X)r_F^{(3)} = 0$ for X vertical iff $\nabla = \nabla^0$ follows from proposition A.6. Then (55) follows by a lengthy but straight forward induction on the total degree using the recursion formulas (46). \square

Corollary 3.10 *Let ∇_0 be a torsion-free connection for Q and ∇^0 the corresponding homogeneous, symplectic, and torsion-free connection for T^*Q . Then the corresponding Fedosov derivation \mathcal{D}_F satisfies*

$$\mathcal{D}_F \pi^* = \pi^* \mathcal{D}_0 \quad (56)$$

and hence the Fedosov-Taylor series of the pull-back of functions $\chi \in C^\infty(Q)[[\lambda]]$ is just the pull-back of their Taylor series with respect to ∇_0

$$\tau_F(\pi^* \chi) = \pi^* \tau_0(\chi). \quad (57)$$

At last in this section we shall discuss the classical limit of the Fedosov derivation \mathcal{D}_F and the Fedosov-Taylor series τ_F . We define $\mathcal{D}_F^{\text{cl}}$ by setting $\lambda = 0$ in \mathcal{D}_F which is well-defined since in any case $\text{ad}_F(r_F)$ starts with $i\lambda\{r_F, \cdot\}_{\text{fib}}$ and analogously we define τ_F^{cl} . Then clearly for $f, g \in C^\infty(T^*Q)$

$$(\mathcal{D}_F^{\text{cl}})^2 = 0 \quad \text{and} \quad \mathcal{D}_F^{\text{cl}} \tau_F^{\text{cl}}(f) = 0 \quad \text{and} \quad \tau_F^{\text{cl}}(\{f, g\}) = \{\tau_F^{\text{cl}}(f), \tau_F^{\text{cl}}(g)\}_{\text{fib}} \quad (58)$$

which is also true for arbitrary symplectic manifolds (compare [14, Sec. 8]). Now we concentrate again on the particular case where we have chosen a torsion-free connection ∇_0 for Q and the corresponding homogeneous, symplectic, and torsion-free connection ∇^0 for T^*Q . Then for any vertical vector field $X \in \Gamma(T(T^*Q))$ we have

$$i_a(X)\mathcal{D}_F + \mathcal{D}_F i_a(X) = [i_a(X), \mathcal{D}_F] = \mathcal{L}_X - i_s(X) - (dx^i \otimes 1)i_s(\nabla_{\partial_{x^i}}^0 X) \quad (59)$$

which is proved by direct calculation using $i_a(X)r_F = 0$. Note that this equation is also true if \mathcal{D}_F is replaced by $\mathcal{D}_F^{\text{cl}}$ since the right hand side is obviously independent of λ .

Theorem 3.11 *Let ∇_0 be a torsion-free connection for Q and ∇^0 the corresponding homogeneous, symplectic and torsion-free connection for T^*Q . Moreover let τ_F be the corresponding Fedosov-Taylor series and τ_F^{cl} the classical part of τ_F and let $q_0 \in Q$. If q^1, \dots, q^n are normal (geodesic) coordinates around q_0 with respect to ∇_0 then for any $\alpha_{q_0} \in T_{q_0}^*Q$ the induced bundle coordinates are normal Darboux coordinates (see e. g. [16, Sec. 2.5, p. 67] for definition) around α_{q_0} with respect to ∇^0 and*

$$\tau_F^{\text{cl}}(f) \Big|_{\alpha_{q_0}} = \sum_{r=0}^{\infty} \frac{1}{r!} \frac{\partial^r f}{\partial x^{i_1} \dots \partial x^{i_r}} \Big|_{\alpha_{q_0}} dx^{i_1} \vee \dots \vee dx^{i_r} \quad (60)$$

*for any $f \in C^\infty(T^*Q)$ where $(x^1, \dots, x^{2n}) = (q^1, \dots, q^n, p_1, \dots, p_n)$.*

PROOF: The fact that the bundle coordinates are normal Darboux coordinates is proved in lemma A.7. First we notice that it is sufficient to prove (60) for functions polynomial in the momenta only. For those functions we shall prove the theorem by induction on the order k in the momenta. For $k = 0$ the statement is true due to corollary 3.10 and theorem 2.5 since the local expression of $\pi^* \tau_0(\cdot)$ in normal (Darboux) coordinates are clearly given by (60) due to lemma A.9. Hence let $f \in C_{pp,k}^\infty(T^*Q)$. We prove the theorem by a local argument: For the coordinate function $\pi^* q^i$ we have $\tau_F^{\text{cl}}(\pi^* q^i)|_{\alpha_{q_0}} = (q^i + dq^i)|_{\alpha_{q_0}} = dq^i$ which implies by (58) and the fact that q^i is a Hamiltonian for the Hamiltonian vector field $-\partial_{p_i}$ that $\tau_F^{\text{cl}}(\frac{\partial f}{\partial p_i})|_{\alpha_{q_0}} = i_s(\partial_{p_i})\tau_F^{\text{cl}}(f)|_{\alpha_{q_0}}$. On the other hand we compute $\mathcal{L}_{\partial_{p_i}}\tau_F^{\text{cl}}(f)$ at α_{q_0} using (59) and (58) which implies $(\mathcal{L}_{\partial_{p_i}}\tau_F^{\text{cl}}(f))|_{\alpha_{q_0}} = i_s(\partial_{p_i})\tau_F^{\text{cl}}(f)|_{\alpha_{q_0}}$ since $\nabla_X^0 \partial_{p_i}|_{\alpha_{q_0}} = 0$ for all $X \in \Gamma(T(T^*Q))$. The homogeneity of τ_F implies $\mathcal{L}_\xi \tau_F^{\text{cl}} = \tau_F^{\text{cl}} \mathcal{L}_\xi$ and hence $\tau_F^{\text{cl}}(f)$ is a polynomial in p_i and dp_i of maximal degree k . Then the last consideration implies that at α_{q_0} it only depends on the combination $p_i + dp_i$. Now using (60) for $\frac{\partial f}{\partial p_i} \in C_{pp,k-1}^\infty(T^*Q)$ due to lemma A.2 the induction is easily finished. \square

4 Homogeneous Fedosov star products of standard ordered type

In this section we shall construct a Fedosov star product of standard ordered type. From now on we shall only use the connection ∇^0 on T^*Q corresponding to a connection ∇_0 on Q as given in (A.4). We define the fibrewise standard ordered product for $a, b \in \mathcal{W} \otimes \Lambda$ by

$$a \circ_s b := \mu \circ e^{\frac{\lambda}{i} i_s(\partial_{p_k}) \otimes i_s(\partial_{q^k}^h)} a \otimes b \quad (61)$$

where $\partial_{q^k}^h = \partial_{q^k} + (\pi^* \Gamma_{kr}^l) p_l \partial_{p_r}$ is the horizontal lift of ∂_{q^k} viewed as vector field on Q with respect to ∇_0 to a vector field on T^*Q (see (130)). Clearly \circ_s is globally defined and an associative deformation of μ which is of the form (25). Moreover we define the maps

$$\Delta_{\text{fib}} := i_s(\partial_{p_k}) i_s(\partial_{q^k}^h) \quad \text{and} \quad \mathcal{S} := e^{\frac{\lambda}{2i} \Delta_{\text{fib}}} \quad (62)$$

which are again globally defined. Then the well-known equivalence (see e. g. [1]) of the standard ordered product and the Weyl product in \mathbb{R}^{2n} can easily be transferred to a fibrewise equivalence: For $a, b \in \mathcal{W} \otimes \Lambda$ we have

$$\mathcal{S}(a \circ_{\text{F}} b) = (\mathcal{S}a) \circ_s (\mathcal{S}b) \quad (63)$$

and moreover, \mathcal{S} commutes with \mathcal{H} and Deg and thus \mathcal{H} is a derivation of \circ_s and \circ_s is homogeneous, too. For the supercommutators using \circ_s we have $\text{ad}_s(a) = \mathcal{S} \circ \text{ad}_{\text{F}}(\mathcal{S}^{-1}a) \circ \mathcal{S}^{-1}$ and $\text{ad}_s(a) = 0$ iff $\text{ad}_{\text{F}}(a) = 0$ iff $\text{deg}_s a = 0$. Moreover we have $[\delta, \Delta_{\text{fib}}] = [\delta, \mathcal{S}] = [\delta, \mathcal{S}^{-1}] = 0$. By a direct calculation of the commutator $\mathcal{B} := \frac{i\lambda}{2} [\nabla^0, \Delta_{\text{fib}}]$ we obtain the following local expression for \mathcal{B}

$$\mathcal{B} = (1 \otimes dq^i) \frac{i\lambda}{3} p_l \left(\pi^* R_{jik}^l \right) i_s(\partial_{p_j}) i_s(\partial_{p_k}) \quad (64)$$

where R_{jik}^l are the components of the curvature tensor of ∇_0 . Now we conjugate the \circ_{F} -superderivation ∇^0 with \mathcal{S} to obtain a \circ_s -superderivation:

$$\mathcal{S} \nabla^0 \mathcal{S}^{-1} = \nabla^0 + \mathcal{B} \quad (65)$$

Note that no higher terms occur since \mathcal{B} already commutes with Δ_{fib} . Now $\nabla^0 + \mathcal{B}$ is a superderivation of \circ_s of antisymmetric degree 1 and total degree 0 which commutes again with \mathcal{H} .

Now we shall follow two ways to obtain a Fedosov derivation for \circ_s : Firstly we just conjugate \mathcal{D}_{F} by \mathcal{S} which will lead indeed to a Fedosov derivation for \circ_s and secondly we start the recursion new with $\nabla^0 + \mathcal{B}$ as term of total degree 0 as in theorem 2.1. The following proposition is straight forward:

Proposition 4.1 *The map $\mathcal{D}' : \mathcal{W} \otimes \Lambda \rightarrow \mathcal{W} \otimes \Lambda$ defined by*

$$\mathcal{D}' := \mathcal{S} \mathcal{D}_{\text{F}} \mathcal{S}^{-1} \quad (66)$$

is a superderivation of antisymmetric degree 1 of \circ_s which has square zero: $\mathcal{D}'^2 = 0$. It commutes with \mathcal{H} and we have

$$\mathcal{D}' = -\delta + \nabla^0 + \mathcal{B} + \frac{\lambda}{i} \text{ad}_s(\mathcal{S} r_{\text{F}}) \quad (67)$$

*where $\mathcal{S} r_{\text{F}} = r_{\text{F}} + \frac{\lambda}{2i} \Delta_{\text{fib}} r_{\text{F}}$. Hence \mathcal{D}' satisfies the conditions of theorem 2.2 and the corresponding Fedosov-Taylor series τ' commutes with \mathcal{H} , too. Moreover for $f, g \in C^\infty(T^*M)[[\lambda]]$*

$$f *' g := \sigma(\tau'(f) \circ_s \tau'(g)) \quad (68)$$

*defines a homogeneous star product for T^*Q .*

Moreover τ' satisfies a recursion formula analogously to the recursion formula (48) for τ_F with the modification that ∇^0 is replaced by $\nabla^0 + \mathcal{B}$, ad_F by ad_S and r_F by $\mathcal{S}r_F$.

Now the star product $*$ ' is no longer of Weyl type but we have the following analogue to the usual standard ordered product in \mathbb{R}^{2n} :

Definition 4.2 *A star product $*$ for a cotangent bundle $\pi : T^*Q \rightarrow Q$ is called of standard ordered type iff for all $\chi \in C^\infty(Q)[[\lambda]]$ and all $f \in C^\infty(T^*Q)[[\lambda]]$*

$$(\pi^*\chi) * f = (\pi^*\chi)f. \quad (69)$$

Then the standard ordered star products can be characterized by the following easy proposition:

Proposition 4.3 *Let $*$ be a star product for a cotangent bundle $\pi : T^*Q \rightarrow Q$ written as $f * g = \sum_{r=0}^\infty \lambda^r M_r(f, g)$ with bidifferential operators M_r then the following statements are equivalent:*

i.) $*$ is of standard ordered type.

ii.) For all $f, g \in C^\infty(T^*Q)[[\lambda]]$ and $\chi \in C^\infty(Q)[[\lambda]]$ we have $((\pi^*\chi)f) * g = (\pi^*\chi)(f * g)$.

iii.) In any local bundle chart the bidifferential operators M_r , $r \geq 1$ are of the form

$$M_r(f, g) = \sum_{l, s, t} M_{i_1 \dots i_s k_1, \dots, k_t}^{j_1 \dots j_l} \frac{\partial^s f}{\partial p_{i_1} \dots \partial p_{i_s}} \frac{\partial^{l+t} g}{\partial q^{j_1} \dots \partial q^{j_l} \partial p_{k_1} \dots \partial p_{k_t}}. \quad (70)$$

The following corollary is immediately checked using the obvious fact that $\tau'\pi^* = \pi^*i^*\tau'\pi^*$ and the particular form of \circ_S and lemma A.2:

Corollary 4.4 *On every cotangent bundle there exists a homogeneous star product of standard ordered type namely the homogeneous Fedosov star product $*$ '.*

Now we shall discuss the other important alternative: starting the recursion new with $\nabla^0 + \mathcal{B}$ as term of total degree 0. First we have to show that $\nabla^0 + \mathcal{B}$ satisfies indeed the conditions of theorem 2.1. Clearly $[\delta, \nabla^0 + \mathcal{B}] = 0$ and moreover

$$\frac{1}{2} [\nabla^0 + \mathcal{B}, \nabla^0 + \mathcal{B}] = \mathcal{S}\nabla^0 \mathcal{S}^{-1} \mathcal{S}\nabla^0 \mathcal{S}^{-1} = \mathcal{S} \frac{i}{\lambda} \text{ad}_F(R) \mathcal{S}^{-1} = \frac{i}{\lambda} \text{ad}_S(\mathcal{S}R) = \frac{i}{\lambda} \text{ad}_S(R)$$

since $\mathcal{S}R = R + \frac{\lambda}{2i} \Delta_{\text{fib}} R$ and $\deg_s \Delta_{\text{fib}} R = 0$ which implies $\text{ad}_S(\Delta_{\text{fib}} R) = 0$. Hence $(\nabla^0 + \mathcal{B})^2$ is an inner superderivation with the element R . It remains to show that $(\nabla^0 + \mathcal{B})R = 0$ but this is clear since $\mathcal{L}_\xi R = R$ and thus $\mathcal{B}R = 0$ due to lemma A.2 and (64) and $\nabla^0 R = 0$ anyway. Thus we can apply indeed theorem 2.1 and obtain the following proposition completely analogously to proposition 4.1 and corollary 4.4:

Proposition 4.5 *There exists a unique element $r_S \in \mathcal{W} \otimes \Lambda^1$ such that $\delta r_S = R + (\nabla^0 + \mathcal{B})r_S + \frac{i}{\lambda} r_S \circ_S r_S$ and $\delta^{-1} r_S = 0$. Moreover $\mathcal{H}r_S = r_S$. Then the corresponding Fedosov derivation*

$$\mathcal{D}_S = -\delta + \nabla^0 + \mathcal{B} + \frac{i}{\lambda} \text{ad}_S(r_S) \quad (71)$$

has square zero and \mathcal{D}_S as well as the corresponding Fedosov-Taylor series τ_S commute with \mathcal{H} . Then

$$f *_S g := \sigma(\tau_S(f) \circ_S \tau_S(g)) \quad (72)$$

where $f, g \in C^\infty(T^*Q)[[\lambda]]$ defines a homogeneous star product of standard ordered type.

Again we have a recursion formula for $\tau_S(f)$ analogously to (48) resp. (30) with the obvious modifications. We shall refer to $*_S$ as the homogeneous Fedosov star product of standard ordered type. At last we consider the element r_S more closely and notice first that $\mathcal{B}r_S = 0$ which follows immediately from the local expression (64) for \mathcal{B} and $\mathcal{H}r_S = r_S$ and lemma A.2. This implies that the recursion formula for r_S as proposed by theorem 2.1 can be simplified to

$$r_S^{(k+3)} = \delta^{-1} \left(\nabla^0 r_S^{(k+2)} - \frac{1}{2} \sum_{l=1}^{k-1} \left\{ r_S^{(l+2)}, r_S^{(k-l+2)} \right\}_{\text{fib}} \right)$$

since $\mathcal{H}r_S = r_S$ and using the explicit expression for \circ_S and lemma A.2. Thus r_S satisfies the same recursion formula as r_F with the same first term $r_S^{(3)} = \delta^{-1} R = r_F^{(3)}$ which implies that they coincide:

Lemma 4.6 *Let r_S be given as in proposition 4.5 then*

$$r_S = r_F. \quad (73)$$

Finally we compute the Fedosov-Taylor series τ' and τ_S of the pull-back of a function on Q :

Proposition 4.7 *Let $\chi \in C^\infty(Q)[[\lambda]]$ and \mathcal{D}' , τ' and \mathcal{D}_S , τ_S be given as above. Then*

$$\mathcal{D}'\pi^* = \mathcal{D}_S\pi^* = \pi^*\mathcal{D}_0 \quad (74)$$

$$\tau'(\pi^*\chi) = \tau_S(\pi^*\chi) = \pi^*\tau_0(\chi) \quad (75)$$

where \mathcal{D}_0 as in (34) and τ_0 is the formal Taylor series with respect to ∇_0 as in (35).

PROOF: Clearly $\mathcal{B}\pi^* = 0$ and $\text{ad}_S(\mathcal{S}r_F)$ as well as $\text{ad}_S(r_F)$ applied to pull-backs by π^* reduce to fibrewise Poisson brackets due to lemma A.2 since $\mathcal{L}_\xi r_F = r_F$. This implies $\mathcal{D}'\pi^* = \mathcal{D}_S\pi^* = \mathcal{D}_F\pi^*$ and by (56) the first equation is proved which implies the second equation. \square

5 Equivalence of $*_F$, $*'$, and $*_S$

In this section we shall construct equivalence transformations of the star products $*_F$, $*'$ and $*_S$. Though DeWilde and Lecomte had shown in [10, Proposition 4.2.] that any homogeneous star products of *Weyl type* on a cotangent bundle are equivalent this is in so far a non-trivial problem since the star products $*'$ and $*_S$ are evidently not of Weyl type. First we define for $f \in C^\infty(T^*Q)[[\lambda]]$

$$Tf := \sigma(\mathcal{S}^{-1}\tau'(f)) \quad (76)$$

then the fibrewise equivalence of \circ_F and \circ_S induces via T an equivalence of $*_F$ and $*'$:

Proposition 5.1 *The map T is homogeneous $[\mathcal{H}, T] = 0$ and an equivalence transformation between $*'$ and $*_F$*

$$T(f *' g) = Tf *_F Tg \quad (77)$$

and we have $\tau_F(Tf) = \mathcal{S}^{-1}\tau'(f)$ and $T^{-1}f = \sigma(\mathcal{S}\tau_F(f))$ where $f, g \in C^\infty(T^*Q)[[\lambda]]$. Moreover T can be written as formal series $T = \text{id} + \sum_{r=1}^{\infty} \lambda^r T_r$ where T_r is a differential operator of order $2r$ given by

$$T_r f = \sum_{s=0}^r \frac{1}{s!} \left(\frac{i}{2} \right)^s \sigma \left(\Delta_{\text{fib}}^s \tau'(f)_{2s}^{(2r)} \right) \quad (78)$$

where $\tau'(f)_l^{(k)}$ denotes the term of total degree k and symmetric degree l .

PROOF: The first part follows from proposition 2.3 and the order of differentiation follows from the fact that $\tau'(f)^{(2r)}$ is a differential operator of order $2r$ which follows directly from the recursion formula (30) for τ' . \square

The construction of an equivalence transformation between $*$ ' and $*$ _S is more involved. First we prove that the Fedosov derivations \mathcal{D}' and \mathcal{D}_S are conjugated to each other by a fibrewise automorphism of \circ_S :

Theorem 5.2 *Let \mathcal{D}_S and \mathcal{D}' be the Fedosov derivations constructed as in section 4 then there exists an element $h \in \mathcal{W}(Q)_{\text{cl}}$ such that*

$$\mathcal{D}_S = e^{\text{ad}_S(\pi^*h)} \mathcal{D}' e^{-\text{ad}_S(\pi^*h)} = e^{\text{ad}_S(\pi^*h)} \mathcal{S} \mathcal{D}_F \mathcal{S}^{-1} e^{-\text{ad}_S(\pi^*h)}. \quad (79)$$

PROOF: Let $h \in \mathcal{W}(Q)_{\text{cl}}$ be an arbitrary element then by direct calculation using the properties of \circ_S and lemma A.2 we obtain

$$\begin{aligned} e^{\text{ad}_S(\pi^*h)}(-\delta)e^{-\text{ad}_S(\pi^*h)} &= -\delta + \text{ad}_S(\pi^*(\delta_0 h)) \\ e^{\text{ad}_S(\pi^*h)}(\nabla^0 + \mathcal{B})e^{-\text{ad}_S(\pi^*h)} &= \nabla^0 + \mathcal{B} - \text{ad}_S(\pi^* \nabla_0 h) \\ e^{\text{ad}_S(\pi^*h)} \frac{i}{\lambda} \text{ad}_S(\mathcal{S} r_F) e^{-\text{ad}_S(\pi^*h)} &= \frac{i}{\lambda} \text{ad}_S \left(r_F - \frac{\lambda}{i} \pi^*(i_s(\varrho_0)h) + \frac{\lambda}{2i} \pi^*(\text{tr} \varrho_0) \right) \end{aligned}$$

where $\text{tr} := i_s(\partial_{q^k})i(dq^i)$ and ϱ_0 is given as in theorem 2.4. Collecting these results we obtain due to $r_S = r_F$

$$e^{\text{ad}_S(\pi^*h)} \mathcal{D}' e^{-\text{ad}_S(\pi^*h)} = \mathcal{D}_S - \text{ad}_S \left(\pi^* \left(\mathcal{D}_0 h - \frac{1}{2} \text{tr} \varrho_0 \right) \right)$$

where \mathcal{D}_0 as in theorem 2.4 applied for ∇_0 on Q . Hence we have to find an element h such that the second term vanishes which is the case iff we find a one-form $\alpha \in \Gamma(\wedge^1 T^*M)$ such that

$$\mathcal{D}_0 h = \frac{1}{2} (\text{tr} \varrho_0 + 1 \otimes \alpha) \quad (80)$$

since in this case $\mathcal{D}_0 h - \frac{1}{2} \text{tr} \varrho_0$ is a central element. Note that we only consider elements $h \in \mathcal{W}_{\text{cl}}$. A necessary condition for (80) to have a solution is that the right hand side is \mathcal{D}_0 -closed since $\mathcal{D}_0^2 = 0$. We have $\mathcal{D}_0 \varrho_0 = R_0$ due to theorem 2.4 and apply tr on both sides leading to $\text{tr} \mathcal{D}_0 \varrho_0 = \text{tr} R_0$ where R_0 is the curvature tensor of ∇_0 . Now a straight forward computation leads to $\text{tr} \mathcal{D}_0 \varrho_0 = \mathcal{D}_0 \text{tr} \varrho_0$. On the other hand if $\alpha \in \Gamma(\wedge^1 T^*Q)$ then $\mathcal{D}_0(1 \otimes \alpha) = 1 \otimes d\alpha$ since ∇_0 is torsion-free. Since the trace of the curvature tensor R_0 is an exact two-form (see lemma A.8) we find always a one-form α such that $\text{tr} R_0 = -d\alpha$ and hence $\mathcal{D}_0(\text{tr} \varrho_0 + 1 \otimes \alpha) = 0$ iff α satisfies $\text{tr} R_0 = -d\alpha$ and thus the necessary condition is fulfilled. But this is also sufficient since the \mathcal{D}_0 -cohomology is trivial on forms: Indeed we define $\mathcal{W} \otimes \Lambda^+ := \{a \in \mathcal{W} \otimes \Lambda \mid \sigma(a) = 0\}$ then we have the following lemma (see [16, Theorem 5.2.5]):

Lemma 5.3 *Let $\mathcal{D}_0^{-1} : \mathcal{W} \otimes \Lambda(Q) \rightarrow \mathcal{W} \otimes \Lambda(Q)$ be defined by*

$$\mathcal{D}_0^{-1} := -\delta_0^{-1} \frac{1}{1 - [\delta_0^{-1}, \nabla_0 + i_s(\varrho_0)]} \quad (81)$$

then for any $a \in \mathcal{W} \otimes \Lambda^+(Q)$ the following ‘deformed Hodge decomposition’ holds

$$\mathcal{D}_0 \mathcal{D}_0^{-1} a + \mathcal{D}_0^{-1} \mathcal{D}_0 a = a \quad (82)$$

and $[\delta_0^{-1}, \nabla_0 + i_s(\varrho_0)]$ commutes with δ_0^{-1} and \mathcal{D}_0 and $\overline{\mathcal{D}_0^{-1} a} = \mathcal{D}_0^{-1} \bar{a}$.

Note that \mathcal{D}_0^{-1} is a well-defined formal series in the symmetric degree. Now choose a one-form α with $\text{tr} R_0 = -d\alpha$. Then we define

$$h := \frac{1}{2} \mathcal{D}_0^{-1} (\text{tr} \varrho_0 + 1 \otimes \alpha) \quad (83)$$

and notice that $\sigma(h) = 0$ since \mathcal{D}_0^{-1} raises the symmetric degree. Moreover $\sigma(\text{tr} \varrho_0 + 1 \otimes \alpha) = 0$ hence we can apply the lemma and find $\mathcal{D}_0 h = \frac{1}{2} (\text{tr} \varrho_0 + 1 \otimes \alpha)$ and thus the theorem is proved. \square

Corollary 5.4 *If h is a solution of (80) for a fixed one-form α then every other solution h' is obtained by $h' = h + \tau_0(\varphi)$ with $\varphi \in C^\infty(Q)$. For a fixed one-form α satisfying $\text{tr} R_0 = -d\alpha$ there exists a unique solution of (80) with $\sigma(h) = \varphi$ for any $\varphi \in C^\infty(Q)$ namely $h = \frac{1}{2}\mathcal{D}_0^{-1}(\text{tr} \varrho_0 + 1 \otimes \alpha) + \tau_0(\varphi)$. The one-form α is determined up to a closed one-form and can be chosen to be real $\bar{\alpha} = \alpha$ which leads to a real $h = \bar{h}$ iff φ is real.*

Corollary 5.5 *If the connection ∇_0 is unimodular then there exists a canonical solution h of (79) uniquely determined by $\mathcal{D}_0 h = \frac{1}{2}\text{tr} \varrho_0$ and $\sigma(h) = 0$ namely $h = \frac{1}{2}\mathcal{D}_0^{-1}\text{tr} \varrho_0$. In this case $h = \bar{h}$ is real.*

Corollary 5.6 *Let h be an arbitrary solution of (80) then $e^{\text{ads}(\pi^* h)}$ is a fibrewise automorphism of \circ_S satisfying (79) and commuting with \mathcal{H} .*

Now we can use the fibrewise automorphism $e^{\text{ads}(\pi^* h)}$ to construct an equivalence transformation between $*_S$ and $*'$ analogously to the construction of T in (76). We define for $f \in C^\infty(T^*Q)[[\lambda]]$

$$Vf := \sigma \left(e^{-\text{ads}(\pi^* h)} \tau_S(f) \right) \quad (84)$$

and get the following proposition completely analogously to proposition 5.1:

Proposition 5.7 *For any solution h of (79) the map V is homogeneous $[\mathcal{H}, V] = 0$ and an equivalence transformation between $*_S$ and $*'$:*

$$V(f *_S g) = Vf *_' Vg \quad (85)$$

and we have $\tau'(Vf) = e^{-\text{ads}(\pi^* h)} \tau_S(f)$ and $V^{-1}f = \sigma(e^{\text{ad}(\pi^* h)} \tau'(f))$ where $f, g \in C^\infty(T^*Q)[[\lambda]]$.

To compute the orders of differentiation in V we first need the following lemma obtaining by the way that $*_S$ is a Vey product:

Lemma 5.8 *Let τ_S be the Fedosov-Taylor series as in proposition 4.5 and let $\tau_S(\cdot)_l^{(k)}$ be the term of total degree k and symmetric degree l . Then $\tau_S(\cdot)_{2s}^{(2r)}$ resp. $\tau_S(\cdot)_{2s+1}^{(2r+1)}$ is a differential operator of order $r + s$ resp. $r + s + 1$ for all $r, s \in \mathbb{N}$. Moreover this implies that the homogeneous Fedosov star product of standard ordered type $*_S$ is a Vey product.*

PROOF: This lemma is proved by a straight forward induction on the total degree using the recursion formula (30) for τ_S . Then the Vey type property of $*_S$ follows directly. \square

Corollary 5.9 *For any solution h of (80) the equivalence transformation V can be written as formal series $V = \text{id} + \sum_{r=1}^{\infty} \lambda^r V_r$ where V_r is a differential operator of order r .*

6 The standard representation

Now we shall construct a canonical representation of the fibrewise algebra (\mathcal{W}, \circ_S) and of the star product algebra $(C^\infty(T^*Q)[[\lambda]], *_S)$ reproducing the well-known standard order quantization rule for cotangent bundles. First we define the representation space

$$\mathfrak{H} := \mathcal{W}(Q) \quad (86)$$

and define the fibrewise standard ordered representation for $a \in \mathcal{W}$ and $\Psi \in \mathfrak{H}$ by

$$\tilde{\varrho}_S(a)\Psi := i^*(a \circ_S \pi^* \Psi) \quad (87)$$

and notice that $\tilde{\varrho}_S : \mathcal{W} \rightarrow \text{End}(\mathfrak{H})$ is indeed a representation of \mathcal{W} with respect to \circ_S :

Lemma 6.1 *Let \circ_S be the fibrewise standard ordered product and $\tilde{\varrho}_S$ be defined as in (87) then $\tilde{\varrho}_S$ is a \circ_S -representation of \mathcal{W} on \mathfrak{H} , i. e. for $a, b \in \mathcal{W}$*

$$\tilde{\varrho}_S(a \circ_S b) = \tilde{\varrho}_S(a) \tilde{\varrho}_S(b). \quad (88)$$

Furthermore

$$\tilde{\varrho}_F(a) := \tilde{\varrho}_S(\mathcal{S}a) \quad (89)$$

defines a representation with respect to the fibrewise Weyl product \circ_F of \mathcal{W} on \mathfrak{H} given by

$$\tilde{\varrho}_F(a) \Psi = i^* \mathcal{S} (a \circ_F \pi^* \Psi). \quad (90)$$

PROOF: The $\mathbb{C}[[\lambda]]$ -linearity of $\tilde{\varrho}_S$ is obvious and the representation property is proved by straight forward computation. Then the fibrewise equivalence (63) of \circ_S and \circ_F implies that $\tilde{\varrho}_F$ is a representation with respect to \circ_F . \square

Now we shall construct a representation ϱ_S for the star product $*_S$ induced by $\tilde{\varrho}_S$. First we notice that the restriction of $\tilde{\varrho}_S$ to $\ker \mathcal{D}_S \cap \mathcal{W}$ is still a representation of $\ker \mathcal{D}_S \cap \mathcal{W}$ and hence it induces via τ_S a representation of $C^\infty(T^*Q)[[\lambda]]$ on \mathfrak{H} . But we have in mind to construct a representation of $\ker \mathcal{D}_S \cap \mathcal{W}$ on the smaller representation space $i^*(\ker \mathcal{D}_S \cap \pi^* \mathfrak{H}) \subset \mathfrak{H}$ which is via τ_0 in bijection with $C^\infty(Q)[[\lambda]]$. In the case of the standard ordered product $*_S$ this can be done directly:

Theorem 6.2 *Let \mathcal{D}_S be the Fedosov derivation constructed as in proposition 4.5 and let τ_S be the corresponding Fedosov-Taylor series and $*_S$ the homogeneous Fedosov star product of standard ordered type. Then*

$$\mathcal{D}_S \pi^* i^* = \pi^* i^* \mathcal{D}_S \quad (91)$$

and

$$\varrho_S(f) \psi := i^*(f *_S \pi^* \psi) = \sigma(\tilde{\varrho}_S(\tau_S(f)) \tau_0(\psi)) \quad (92)$$

where $f \in C^\infty(T^*Q)[[\lambda]]$ and $\psi \in C^\infty(Q)[[\lambda]]$ defines a representation of $C^\infty(T^*Q)[[\lambda]]$ with respect to $*_S$ on $C^\infty(Q)[[\lambda]]$.

PROOF: One immediately checks that δ , ∇^0 and \mathcal{B} commute with $\pi^* i^*$. Moreover $\text{ad}_S(r_S)$ commutes with $\pi^* i^*$ too due to the particular form of \circ_S and $\mathcal{L}_\xi r_S = r_S$ and lemma A.2. Hence (91) is shown. This ensures that ϱ_S defines indeed a representation which is now a straight forward computation using (88) and proposition 4.7. \square

Note that (91) is crucial for the representation property of ϱ_S and that neither \mathcal{D}_F nor \mathcal{D}' commute with $\pi^* i^*$. We shall refer to ϱ_S as *standard representation* with respect to $*_S$. A representation with respect to the homogeneous Fedosov star product of Weyl type $*_F$ can be constructed by use of the equivalence transformations V and T but we shall see in the next section that there is another star product $*_W$ of Weyl type with a corresponding representation. Now we compute an explicit formula for the representation ϱ_S and rediscover the well-known standard order quantization rule for cotangent bundles (see (19)):

Theorem 6.3 *Let $*_S$ be the homogeneous Fedosov star product of standard ordered type and let ϱ_S be the corresponding standard representation. Then for $f \in C^\infty(T^*Q)[[\lambda]]$ and $\psi \in C^\infty(Q)[[\lambda]]$ we have*

$$\varrho_S(f) \psi = \sum_{r=0}^{\infty} \frac{1}{r!} \left(\frac{\lambda}{i} \right)^r i^* \left(\frac{\partial^r f}{\partial p_{i_1} \cdots \partial p_{i_r}} \right) i_s(\partial_{q^{i_1}}) \cdots i_s(\partial_{q^{i_r}}) D_0^{(r)} \psi \quad (93)$$

where $D_0^{(r)}\psi := \frac{1}{r!}(dq^k \vee \nabla_0 \partial_{q^k})^r \psi$ is the r th symmetrized covariant derivative of ψ with respect to ∇_0 . In particular for a function $f \in C_{pp,k}^\infty(T^*Q)$ polynomial in the momenta of order k we have

$$\varrho_S(f)\psi = \frac{1}{k!} \left(\frac{\lambda}{i} \right)^k \langle F, D_0^{(r)}\psi \rangle \quad (94)$$

where $F \in \Gamma(\bigvee^k TQ)$ is the symmetric tensor field such that $f = \widehat{F}$ and $\langle \cdot, \cdot \rangle$ denotes the natural pairing. If \widehat{X} is linear in the momentum where $X \in \Gamma(TQ)$ then

$$\varrho_S(f)\psi = \frac{\lambda}{i} \mathcal{L}_X \psi. \quad (95)$$

PROOF: We compute $\varrho_S(f)\psi$ using (92) and proposition 4.7 and obtain

$$\varrho_S(f)\psi = \sum_{r=0}^{\infty} \frac{1}{r!} \left(\frac{\lambda}{i} \right)^r i^* \sigma(i_s(\partial_{p_{i_1}}) \cdots i_s(\partial_{p_{i_r}}) \tau_S(f)) \sigma(i_s(\partial_{q^{i_1}}) \cdots i_s(\partial_{q^{i_r}}) \tau_0(\psi)).$$

But since τ_0 is the Taylor series with respect to ∇_0 we find that the last part $\sigma(i_s(\partial_{q^{i_1}}) \cdots i_s(\partial_{q^{i_r}}) \tau_0(\psi))$ equals $i_s(\partial_{q^{i_1}}) \cdots i_s(\partial_{q^{i_r}}) D_0^{(r)}\psi$ and thus we only have to prove that $i^* \sigma(i_s(\partial_{p_{i_1}}) \cdots i_s(\partial_{p_{i_r}}) \tau_S(f))$ coincides with $i^* (\frac{\partial^r f}{\partial p_{i_1} \cdots \partial p_{i_r}})$. But this follows directly from the following lemma: \square

Lemma 6.4 *Let $f \in C^\infty(T^*Q)$ then in any local bundle chart there exist locally defined elements $\tilde{\tau}_i^r(f)$ such that for any total degree $r \geq 0$*

$$\tau_S(f)^{(r)} = D^{(r)}f + dq^i \vee \tilde{\tau}_i^r(f) \quad (96)$$

PROOF: This lemma is proved by a straight forward induction on the total degree using the recursion formulas (30) for τ_S . \square

Corollary 6.5 *The restriction of ϱ_S to the $\mathbb{C}[\lambda]$ -submodule $C_{pp}^\infty(T^*Q)[\lambda]$ as well as the restriction to the $\mathbb{C}[[\lambda]]$ -submodule $C_p^\omega(T^*Q)[[\lambda]]$ is injective.*

7 Two different homogeneous star products of Weyl type

In this section we shall construct an analogue to the operator N mentioned in the introduction. This operator allows us to define a star product of Weyl type $*_W$ equivalent to $*_S$ which corresponds to the Weyl ordering prescription of flat \mathbb{R}^{2n} and turns out to be different form $*_F$ in general.

In this section we sometimes denote the complex conjugation by \mathcal{C} , i. e. $\mathcal{C}a := \bar{a}$ for $a \in \mathcal{W} \otimes \Lambda$ and define

$$\mathcal{C}_S := e^{\text{ad}_S(\pi^*h)} \mathcal{S} \mathcal{C} \mathcal{S}^{-1} e^{-\text{ad}_S(\pi^*h)} \quad (97)$$

where $h \in \mathcal{W}(Q)_{\text{cl}}$ is constructed as in theorem 5.2 with $\sigma(h) = 0$ incorporating a particular but fixed choice of a *real* one-form $\alpha \in \Gamma(\bigwedge^1 T^*Q)$ such that $-d\alpha = \text{tr} R_0$. Then we have the following lemmata:

Lemma 7.1 *Let $a, b \in \mathcal{W} \otimes \Lambda$ with $\deg_a a = ka$ and $\deg_a b = lb$ then*

$$\mathcal{C}(a \circ_F b) = (-1)^{kl} (\mathcal{C}b) \circ_F (\mathcal{C}a) \quad (98)$$

and $[\mathcal{D}_F, \mathcal{C}] = 0$ and

$$\mathcal{C}_S(a \circ_S b) = (-1)^{kl} (\mathcal{C}_S b) \circ_S (\mathcal{C}_S a) \quad (99)$$

$$[\mathcal{D}_S, \mathcal{C}_S] = 0 = [\mathcal{H}, \mathcal{C}_S]. \quad (100)$$

PROOF: Equation (98) and the fact $[\mathcal{D}_F, \mathcal{C}] = 0$ are well-known (see e. g. [5, Lemma 3.3]) and imply together with (79) the other statements of the lemma. \square

Lemma 7.2 *Let $f, g \in C^\infty(T^*Q)[[\lambda]]$ then the map*

$$C_S f := \sigma(\mathcal{C}_S \tau_S(f)) \quad (101)$$

*defines an involutive anti- $\mathbb{C}[[\lambda]]$ -linear anti-automorphism of the star product $*_S$, i. e. $C_S^2 = \text{id}$ and*

$$C_S(f *_S g) = (C_S g) *_S (C_S f) \quad (102)$$

and we have $\tau_S(C_S f) = \mathcal{C}_S \tau_S(f)$ and $[C_S, \mathcal{H}] = 0$. Moreover $C_S \mathcal{C}$ is a formal series of differential operators

$$C_S \mathcal{C} = \text{id} + \sum_{r=1}^{\infty} \lambda^r C_S^{(r)} \quad (103)$$

where $C_S^{(r)}$ is a differential operator of order $2r$ and $C_S^{(1)} = \frac{1}{i} \Delta$ and

$$\Delta := \nabla_{(\partial_{p_i}, \partial_{q_i}^h)}^{0^2} + \nabla_{(\alpha^v)}^0 \quad (104)$$

which is clearly globally defined and takes the following form in a bundle chart

$$\Delta = \partial_{q^i} \partial_{p_i} + p_r \pi^*(\Gamma_{ij}^r) \partial_{p_i} \partial_{p_j} + \pi^*(\Gamma_{ij}^i) \partial_{p_j} + \pi^*(\alpha_j) \partial_{p_j}. \quad (105)$$

PROOF: Equation (102) is proved in a completely analogous manner as proposition 2.3. Since the term $\tau_S(\cdot)^{(k)}$ of total degree k is easily seen to be of order k as a differential operator and since \mathcal{C}_S does obviously not decrease the total degree (\mathcal{S} being of total degree 0 and $\pi^* h$ contains only terms of positive total degree) equation (103) follows. Moreover $C_S^{(1)} = -i\Delta$ follows by straight forward computation using $\tau_S(f)^{(2)} = \frac{1}{2}(dx^k \vee \nabla_{\partial_{x^k}}^0)^2 f$ due to (30) and $h_1 = -\frac{1}{2}\alpha \otimes 1$ where h_1 is the term of symmetric degree 1 in h due to (83). \square

Now we come to a simple analogue of the operator N mentioned in (4):

$$N := \exp\left(\frac{\lambda}{2i}\Delta\right) \quad (106)$$

which clearly is a formal power series in λ of differential operators and $[N, \mathcal{H}] = 0$. Motivated by equation (12) we define an equivalent star product to $*_S$ using N as an equivalence transformation by

$$f *_W g := N^{-1}((Nf) *_S (Ng)) \quad (107)$$

for $f, g \in C^\infty(T^*Q)[[\lambda]]$ which is clearly bidifferential and homogeneous. Furthermore we have the following theorem:

Theorem 7.3 *i.) The operator $N^2 \mathcal{C}$ coincides with C_S .*

*ii.) The star product $*_W$ is a star product of Weyl type where in particular*

$$\overline{f *_W g} = \overline{g} *_W \overline{f} \quad (108)$$

*for $f, g \in C^\infty(T^*Q)[[\lambda]]$.*

*iii.) A representation ϱ_W of $(C^\infty(T^*Q)[[\lambda]], *_W)$ on $C^\infty(Q)[[\lambda]]$ is given by the following analogue of equation (5)*

$$\varrho_W(f) := \varrho_S(Nf). \quad (109)$$

PROOF:

i.) We proceed in two steps: Let us first prove that $N^2\mathcal{C}$ is an involutive anti-linear anti-automorphism of $(C^\infty(T^*Q)[[\lambda]], *_S)$. This is equivalent to the identity

$$N^2\mathcal{C}(f *_S g) - (N^2\mathcal{C}g) *_S (N^2\mathcal{C}f) = 0$$

for all $f, g \in C^\infty(T^*Q)[[\lambda]]$. Since $*_S$ consists in bidifferential operators this identity is an identity of bidifferential operators for each power of λ . Hence it suffices to check it on an arbitrarily small neighbourhood of each point $m \in T^*Q$ on $f, g \in C_{pp}^\infty(T^*Q)$. Since the standard representation ϱ_S is injective on $C_{pp}^\infty(T^*Q)[\lambda]$ it is sufficient to prove this identity after having applied ϱ_S to the left hand side. Now let $q \in Q$ be arbitrary and choose a contractible open neighbourhood U around q which lies in the domain of a chart. Suppose that the supports of f, g lie both in $\pi^{-1}(U)$. Then there is a local volume form μ on U such that $\nabla_{0X}\mu = \alpha(X)\mu$ for each vector field X on U : indeed, choose an arbitrary local volume form μ' on U . Then $\nabla_{0X}\mu' = \alpha'(X)\mu'$ for a certain locally defined one-form α' on U . Since $d\alpha'(X, Y)\mu' = ([\nabla_{0X}, \nabla_{0Y}] - \nabla_{0[X, Y]})\mu' = -(\text{tr}R)(X, Y)\mu' = d\alpha(X, Y)\mu'$ it follows from the Poincaré lemma that there is a local real-valued smooth function ϕ on U such that $\alpha' = \alpha + d\phi$. Then $\mu := e^{-\phi}\mu'$ will clearly do the job. Consider next the space $\mathcal{D}(U)$ of all smooth complex-valued functions on Q whose support lies in U and is compact. This space is an inner product space with respect to the Lebesgue integral $\langle \phi, \psi \rangle_U := \int_U \bar{\phi}\psi\mu$. We define the following covariant divergence operator $\text{div}_\alpha : \Gamma(\bigvee^k TQ) \rightarrow \Gamma(\bigvee^{k-1} TQ)$ with $\Gamma(\bigvee^l TQ) := \{0\}$ for negative integers l :

$$\text{div}_\alpha S := \text{div}S + i_s(\alpha)S \quad \text{where} \quad \text{div}S := i_s(dq^i)\nabla_{\partial_i}S \quad (110)$$

which is clearly globally defined. Let T be in $\Gamma(\bigvee^{k+1} TQ)$. Using the global coordinates (q^1, \dots, q^n) in U and $D_0 := dq^i \vee \nabla_{\partial_i}$ we get for any $\psi \in \mathcal{D}(U)$:

$$\begin{aligned} & \frac{1}{(k+1)!} T^{i_1 \dots i_{k+1}} i_s(\partial_{i_1}) \dots i_s(\partial_{i_{k+1}}) D_0^{k+1} \psi \\ &= \mathcal{L}_{\partial_j} \left(\frac{1}{(k+1)!} T^{i_1 \dots i_{k+1}} i_s(\partial_{i_1}) \dots i_s(\partial_{i_{k+1}}) (dq^j \vee D_0^k \psi) \right) \\ & \quad - \frac{1}{(k+1)!} (\nabla_{0\partial_j} T)^{i_1 \dots i_{k+1}} i_s(\partial_{i_1}) \dots i_s(\partial_{i_{k+1}}) (dq^j \vee D_0^k \psi) \\ & \quad + \frac{1}{(k+1)!} \Gamma_{rj}^r T^{i_1 \dots i_{k+1}} i_s(\partial_{i_1}) \dots i_s(\partial_{i_{k+1}}) (dq^j \vee D_0^k \psi) \\ &= \mathcal{L}_{\partial_j} \left(\frac{1}{k!} T^{j i_1 \dots i_k} i_s(\partial_{i_1}) \dots i_s(\partial_{i_k}) D_0^k \psi \right) \\ & \quad - \frac{1}{k!} (\nabla_{0\partial_j} T)^{j i_1 \dots i_k} i_s(\partial_{i_1}) \dots i_s(\partial_{i_k}) D_0^k \psi \\ & \quad + \frac{1}{k!} \Gamma_{rj}^r T^{j i_1 \dots i_k} i_s(\partial_{i_1}) \dots i_s(\partial_{i_k}) D_0^k \psi. \end{aligned}$$

Since for any vector field X on Q we obviously have $\mathcal{L}_X \mu = (\text{div}_\alpha X)\mu$ it can easily be seen by partial integration and induction that the following identity is true:

$$\begin{aligned} \int_U \bar{\phi} \varrho_S(\hat{T}) \psi \mu &= \int_U \overline{\left(\frac{1}{k!} \left(\frac{\lambda}{i} \right)^k (\text{div}_\alpha)^k (\phi \bar{T}) \right)} \psi \mu \\ &= \int_U \overline{\left(\frac{1}{k!} \left(\frac{\lambda}{i} \right)^k \sum_{s=0}^k \binom{k}{s} ((\text{div}_\alpha)^s \bar{T})^{i_1 \dots i_{k-s}} i_s(\partial_{i_1}) \dots i_s(\partial_{i_{k-s}}) D_0^{k-s} \phi \right)} \psi \mu \end{aligned}$$

$$= \int_U \overline{\left(\varrho_S \left(\sum_{s=0}^k \frac{1}{s!} \left(\frac{\lambda}{i} \right) ((\widehat{\text{div}}_\alpha)^s \overline{T}) \right) \phi \right)} \psi \mu = \int_U \overline{\left(\varrho_S \left(N^2 \widehat{T} \right) \phi \right)} \psi \mu$$

where in the last equality we have used the identity $(\widehat{\text{div}}_\alpha)^s T = \Delta^s \widehat{T}$ for $s \in \mathbb{N}$ which follows from

$$\widehat{\text{div}}_\alpha T = \Delta \widehat{T} \quad (111)$$

which can easily be proved by calculation. It follows that on U the differential operator $\varrho_S(\widehat{T})$ has a formal adjoint $\varrho_S(\widehat{T})^\dagger$ given by $\varrho_S(N^2 \widehat{T})$. By the well-known rules of multiplying adjoints, viz.

$$\left(\varrho_S(\widehat{T}) \varrho_S(\widehat{S}) \right)^\dagger = \varrho_S(\widehat{S})^\dagger \varrho_S(\widehat{T})^\dagger$$

we finally get that $N^2 \mathcal{C}$ is an involutive anti-linear anti-automorphism since clearly $N^2 \mathcal{C} N^2 \mathcal{C} = \text{id}$.

Secondly, since now both C_S and $N^2 \mathcal{C}$ are involutive anti-linear anti-automorphisms the composition $C_S^{-1} N^2 \mathcal{C}$ will be a linear automorphism of the star product \ast_S of standard ordered type which again is a formal series in λ whose coefficients consist of differential operators. Hence it suffices to prove the desired identity $C_S^{-1} N^2 \mathcal{C} = \text{id}$ locally on functions in $C_{pp}^\infty(T^*Q)$. Take again the above neighbourhood U . By proposition 3.7 every such function can be written as a finite linear combination over terms consisting of star products of functions belonging to $C_{pp,0}^\infty(T^*Q)[\lambda] + C_{pp,1}^\infty(T^*Q)[\lambda]$. By the automorphism property it thus suffices to check our identity on these functions at most linear in the momenta. An easy computation using lemma 7.2 shows that this is the case.

ii.) This is straight forward: using the fact that $N^2 \mathcal{C}$ is an involutive anti-linear anti-automorphism of \ast_S it follows that complex conjugation is an involutive anti-linear anti-automorphism of \ast_W . Moreover since all constructions depend only on the combination $\frac{\lambda}{i}$ it follows that the parity transformation generated by $\lambda \mapsto -\lambda$ is an anti-automorphism which shows that \ast_W is indeed of Weyl type.

iii.) This is a straight forward computation. □

Remark: The two star products \ast_W and \ast_F are not the same in general as we shall see evaluating both star products on functions linear in the momenta: Let $X, Y \in \Gamma(TQ)$ and consider $\widehat{X}, \widehat{Y} \in C_{pp,1}^\infty(T^*Q)$. Then one gets by direct calculation $N\widehat{X} = \widehat{X} + \frac{\lambda}{2i} \widehat{\text{div}}_\alpha X$ as well as

$$\Delta(\widehat{X}\widehat{Y}) = \widehat{X} \widehat{\text{div}}_\alpha Y + \widehat{Y} \widehat{\text{div}}_\alpha X + \widehat{\nabla_{0X} Y} + \widehat{\nabla_{0Y} X} \quad (112)$$

$$\Delta^2(\widehat{X}\widehat{Y}) = 2(\widehat{\text{div}}_\alpha X)(\widehat{\text{div}}_\alpha Y) + \mathcal{L}_X(\widehat{\text{div}}_\alpha Y) + \mathcal{L}_Y(\widehat{\text{div}}_\alpha X) + \widehat{\text{div}}_\alpha(\nabla_{0X} Y) + \widehat{\text{div}}_\alpha(\nabla_{0Y} X). \quad (113)$$

Using theorem 6.3 one obtains $\varrho_S(N\widehat{X}) = \frac{\lambda}{i}(\mathcal{L}_X + \frac{1}{2} \widehat{\text{div}}_\alpha X)$ and

$$\varrho_S(\widehat{X}\widehat{Y}) = \left(\frac{\lambda}{i} \right)^2 \mathcal{L}_X \mathcal{L}_Y - \frac{\lambda}{i} \varrho_S(\widehat{\nabla_{0X} Y}). \quad (114)$$

This enables us to calculate $\varrho_S(N\widehat{X})\varrho_S(N\widehat{Y})$ and using the injectivity of ϱ_S on functions polynomial in the momenta (corollary 6.5) and the representation property of ϱ_S we obtain

$$\begin{aligned} & (N\widehat{X}) \ast_S (N\widehat{Y}) \\ &= \widehat{X}\widehat{Y} + \frac{\lambda}{i} \left(\widehat{\nabla_{0X} Y} + \frac{1}{2} \widehat{Y} \widehat{\text{div}}_\alpha X + \frac{1}{2} \widehat{X} \widehat{\text{div}}_\alpha Y \right) + \frac{1}{2} \left(\frac{\lambda}{i} \right)^2 \left(\mathcal{L}_X(\widehat{\text{div}}_\alpha Y) + \frac{1}{2} (\widehat{\text{div}}_\alpha X)(\widehat{\text{div}}_\alpha Y) \right). \end{aligned}$$

Finally we calculate $\widehat{X} \ast_W \widehat{Y} = N^{-1}((N\widehat{X}) \ast_S (N\widehat{Y}))$ and obtain

$$\widehat{X} \ast_W \widehat{Y} = \widehat{X}\widehat{Y} + \frac{i\lambda}{2} \{ \widehat{X}, \widehat{Y} \} + \left(\frac{i\lambda}{2} \right)^2 M_2^W(\widehat{X}, \widehat{Y}) \quad (115)$$

where

$$M_2^w(\widehat{X}, \widehat{Y}) = \frac{1}{2} \left(\mathcal{L}_X(\text{div}_\alpha Y) + \mathcal{L}_Y(\text{div}_\alpha X) - \text{div}_\alpha(\nabla_{0X} Y) - \text{div}_\alpha(\nabla_{0Y} X) \right) \quad (116)$$

observing that $\mathcal{L}_X(\text{div}_\alpha Y) - \mathcal{L}_Y(\text{div}_\alpha X) - \text{div}_\alpha(\nabla_{0X} Y) + \text{div}_\alpha(\nabla_{0Y} X) = 0$ due to $d\alpha(X, Y) = -\text{tr}R_0(X, Y)$. Writing the coordinate expression $X_{|l}^k Y_{|k}^l$ for $\text{trace}(Z \mapsto \nabla_{0\nabla_0 Z} X Y)$ to avoid clumsy notation the operator M_2^w can be simplified to

$$M_2^w(\widehat{X}, \widehat{Y}) = -X_{|l}^k Y_{|k}^l - \frac{1}{2} \left(\text{Ric}_0(X, Y) + \text{Ric}_0(Y, X) - (\nabla_{0X} \alpha)(Y) - (\nabla_{0Y} \alpha)(X) \right) \quad (117)$$

where $_{|k}$ denotes the covariant derivative with respect to ∂_{q^k} and Ric_0 denotes the Ricci tensor of ∇_0 . On the other hand we compute the Fedosov star product $*_F$ of \widehat{X} and \widehat{Y} using [5, Theorem 3.4] and obtain here the following expression for the second order term $M_2^F(\widehat{X}, \widehat{Y})$:

$$M_2^F(\widehat{X}, \widehat{Y}) = -X_{|l}^k Y_{|k}^l. \quad (118)$$

For general manifolds Q with torsion-free connection ∇_0 these two star products do not coincide for any choice of α : for example, let Q be equal to S^2 with the standard metric. Its Levi-Civita connection ∇_0 is clearly unimodular whence every possible α is a closed one-form, hence exact ($\alpha = d\phi$) since the first de Rham cohomology group of the two-sphere vanishes. If the two expressions (117) and (118) were the same for any X, Y then the Ricci tensor would be equal to the second covariant derivative of ϕ . In particular upon contracting with the inverse metric we would obtain that the Laplacian of ϕ were equal to a positive multiple of the scalar curvature of S^2 which is positive and constant. But this differential equation has no smooth solution since the integral (with respect to the Riemannian volume) over S^2 of the Laplacian of ϕ would vanish as opposed to the integral over S^2 of a positive constant.

8 Example: The cotangent bundle of a Lie group

This section shall be dedicated to finding explicit formulae for the homogeneous Fedosov star product $*_s$ of standard ordered type on the cotangent bundle of a connected Lie group G , that is equipped with the natural torsion-free connection defined by $\nabla_{0U} V := \frac{1}{2}[U, V]$ for left-invariant vector fields U, V on G , obtained using the standard representation ϱ_s for functions polynomial in the momentum variables. To express the bidifferential operators defining the star product we make use of the natural set of basis sections in the tangent bundle of T^*G given by Y_i resp. Z^j that are the horizontal resp. vertical lifts of a basis of left-invariant vector fields X_i resp. the dual left-invariant one-forms θ^j with respect to the flat connection on G (see definition A.1) which is defined by $\widetilde{\nabla}_U V = 0$ for left-invariant vector fields U, V . Instead of using Darboux coordinates it will be convenient to use natural fibre-variables P_i ($1 \leq i \leq \dim(G)$) given by $P_i : T^*G \rightarrow \mathbb{R} : \alpha_g \rightarrow \alpha_g(X_i)$ for $\alpha_g \in T_g^*G$. The obtained expression for $*_s$ shall moreover be related to a star product $*_G$ of Weyl type by means of an equivalence transformation N and we shall prove that $*_G$ coincides (up to a rescaling of the formal parameter) with the star product obtained by Gutt in [17] using cohomological methods instead of our purely algebraic approach. As a first step we shall find an expression for the star product $*_s$ of polynomial functions in the momenta on T^*G that are invariant under the canonical lift $T^*(l_g)$ of the left-translations $l_g : G \rightarrow G$ to T^*G . Since those polynomials are generated by functions of the form $\widehat{U^{\vee k}}$ for left-invariant vector fields U on G , we may restrict our calculations to functions

$$e_U : \alpha_g \mapsto \sum_{r=0}^{\infty} \frac{1}{r!} (U(\alpha_g))^r \in C_p^\omega(T^*G)[[\lambda]].$$

Lemma 8.1 *Let U, V be left-invariant vector fields on G then we have the formula [13], [17]*

$$e_U *_S e_V = e_{\frac{i}{\lambda} H(\frac{\lambda}{i} U, \frac{\lambda}{i} V)}, \quad (119)$$

at which H denotes the Baker-Campbell-Hausdorff series. Moreover this equation uniquely determines bidifferential operators M_r^{inv} defined by

$$e_U *_S e_V =: \sum_{r=0}^{\infty} \left(\frac{\lambda}{i} \right)^r M_r^{inv}(e_U, e_V), \quad (120)$$

that are of order r in every argument, homogeneous of degree $-r$ and only containing derivatives with respect to vertical directions and multiplications with polynomials in the momenta on T^*G , that are invariant under $T^*(l_g)$.

PROOF: We just have to notice that $\varrho_S(e_U)\chi$ is given by $\exp(\circ \frac{\lambda}{i} U)\chi$ for $\chi \in C^\infty(G)$, at which \circ denotes the composition of the left-invariant vector fields viewed as differential operators on $C^\infty(G)$. Using this fact yields

$$\begin{aligned} \varrho_S(e_U *_S e_V)\chi &= (\varrho_S(e_U) \circ \varrho_S(e_V))\chi = \left(\exp\left(\circ \frac{\lambda}{i} U\right) \circ \exp\left(\circ \frac{\lambda}{i} V\right) \right) \chi \\ &= \exp\left(\circ \frac{\lambda}{i} \frac{i}{\lambda} H\left(\frac{\lambda}{i} U, \frac{\lambda}{i} V\right)\right) \chi = \varrho_S\left(e_{\frac{i}{\lambda} H(\frac{\lambda}{i} U, \frac{\lambda}{i} V)}\right) \chi. \end{aligned}$$

Now both arguments $e_U *_S e_V$ and $e_{\frac{i}{\lambda} H(\frac{\lambda}{i} U, \frac{\lambda}{i} V)}$ are elements in the submodule $C_p^\omega(T^*Q)[[\lambda]]$ and hence corollary 6.5 ensures that they coincide. The assertions about the bidifferential operators M_r^{inv} are obvious consequences of properties of the Baker-Campbell-Hausdorff series resp. of proposition 4.5. \square

Proposition 8.2 *The star product $*_S$ of two functions $f, g \in C^\infty(T^*G)$ is given by*

$$f *_S g = \sum_{r=0}^{\infty} \left(\frac{\lambda}{i} \right)^r \sum_{t=0}^r \frac{1}{t!} M_{r-t}^{inv}(Z^{j_1} \dots Z^{j_t} f, Y_{j_1} \dots Y_{j_t} g). \quad (121)$$

By lemma 8.1 it is obvious that this star product is of Vey type.

PROOF: First we notice that it is sufficient to prove (121) for functions polynomial in the momenta. Using $\varrho_S(\widehat{S})\chi = \frac{1}{l!} \left(\frac{\lambda}{i} \right)^l S^{j_1 \dots j_l} X_{j_1} \dots X_{j_l} \chi$ for $\widehat{S} \in C_{pp,l}^\infty(T^*G)$ and an analogous formula for $\widehat{T} \in C_{pp,k}^\infty(T^*G)$, which are obtained from $D_0^k \chi = (\theta^l \vee \nabla_{0X_l})^k \chi = X_{i_1} \dots X_{i_k} \chi \theta^{i_1} \vee \dots \vee \theta^{i_k}$ and the symmetry of S resp. T , we get for $\chi \in C^\infty(G)$

$$\begin{aligned} \varrho_S(\widehat{S} *_S \widehat{T})\chi &\stackrel{(a)}{=} \sum_{t=0}^l \frac{1}{k!t!(l-t)!} \left(\frac{\lambda}{i} \right)^t S^{j_1 \dots j_l} (X_{j_1} \dots X_{j_t} T^{i_1 \dots i_k}) \varrho_S(P_{j_{t+1}} \dots P_{j_l} *_S P_{i_1} \dots P_{i_k})\chi \\ &\stackrel{(b)}{=} \varrho_S \left(\sum_{r=0}^{\infty} \left(\frac{\lambda}{i} \right)^r \sum_{t=0}^l \frac{1}{t!(l-t)!} \left(\frac{\lambda}{i} \right)^t M_{r-t}^{inv} \left(\pi^* S^{j_1 \dots j_l} P_{j_{t+1}} \dots P_{j_l}, Y_{j_1} \dots Y_{j_t} \widehat{T} \right) \right) \chi \\ &\stackrel{(c)}{=} \varrho_S \left(\sum_{r=0}^{\infty} \left(\frac{\lambda}{i} \right)^r \sum_{t=0}^r \frac{1}{t!} M_{r-t}^{inv} \left(Z^{j_1} \dots Z^{j_t} \widehat{S}, Y_{j_1} \dots Y_{j_t} \widehat{T} \right) \right) \chi. \end{aligned}$$

Equation (a) is a consequence of the Leibniz rule and $\varrho_S(P_{i_1} \dots P_{i_k})\chi = \left(\frac{\lambda}{i} \right)^k \frac{1}{k!} \sum_{\sigma \in S_k} X_{i_{\sigma(1)}} \dots X_{i_{\sigma(k)}} \chi$. In

(b) we used lemma 8.1 and the fact that $\varrho_S(\pi^* \psi F) = \psi \varrho_S(F)$ for $\psi \in C^\infty(G)$ and the π -relatedness of the

vector fields X_i and Y_i . In (c) the sum over t was extended to ∞ since the terms $Z^{j_1} \dots Z^{j_t} \widehat{S}$ vanish for $t > l$ due to lemma A.2. Then (121) follows by the injectivity of ϱ_S restricted to $C_{pp}^\infty(T^*G)$ proved in corollary 6.5. \square

Now according to the last section we consider a Weyl ordered star product defined by

$$f *_G g := N^{-1} (Nf *_S Ng) \text{ and } N := \exp \left(\frac{\lambda}{2i} \Delta \right) \quad (122)$$

with $\Delta = Y_i Z^i + \frac{1}{2} C_{li}^l Z^i + \pi^*(\alpha_i) Z^i$ (which is easily computed using equation (104)) denoting by α_i the components of a one-form fulfilling $d\alpha = -\text{tr} R_0 = 0$ due to the particular choice of the connection ∇_0 where $C_{ij}^k := \theta^k([X_i, X_j])$ are the structure constants of the Lie algebra \mathfrak{g} of G . Choosing $\alpha = (t - \frac{1}{2}) C_{li}^l \theta^i$ for $t \in \mathbb{R}$ it is easily verified that $d\alpha = 0$ and $N = N_0 A_t$ with $N_0 := \exp \left(\frac{\lambda}{2i} Y_i Z^i \right)$ and $A_t := \exp \left(\frac{\lambda}{2i} t d_S \right)$ with $d_S := C_{li}^l Z^i$, since $Y_i Z^i$ and d_S commute. For any choice of $t \in \mathbb{R}$ we shall prove the coincidence of $*_G$ with the one constructed in [17].

Lemma 8.3 *For $\alpha = (t - \frac{1}{2}) C_{li}^l \theta^i$ ($t \in \mathbb{R}$) and N defined as above we have*

$$e_U *_G e_V = e_U *_S e_V \text{ for all left-invariant vector fields } U, V \text{ on } G. \quad (123)$$

Moreover A_t is a one-parameter-group of automorphisms of $*_S$, i. e. $A_t A_s = A_{t+s}$ and $A_0 = \text{id}$ and

$$A_t (f *_S g) = (A_t f) *_S (A_t g) \quad \forall f, g \in C^\infty(T^*G)[[\lambda]], \forall t \in \mathbb{R}, \quad (124)$$

implying that d_S is a derivation of $*_S$. Moreover this implies that all choices of $t \in \mathbb{R}$ lead to the same star product $*_G$.

PROOF: Since $N = N_0 A_t$ we prove the first assertion in two steps using N_0 and A_t separately. Obviously we have $N_0 e_U = e_U$ and therefore using equation (119)

$$N_0^{-1} ((N_0 e_U) *_S (N_0 e_V)) = N_0^{-1} (e_U *_S e_V) = N_0^{-1} e_{\frac{1}{\lambda} H(\frac{\lambda}{i} U, \frac{\lambda}{i} V)} = e_{\frac{1}{\lambda} H(\frac{\lambda}{i} U, \frac{\lambda}{i} V)} = e_U *_S e_V.$$

For the second part we compute $\varrho_S(A_t e_U) = \exp \left(\frac{\lambda}{2i} t C_{li}^l U^i \right) \varrho_S(e_U)$ yielding

$$\varrho_S((A_t e_U) *_S (A_t e_V)) = \exp \left(\frac{\lambda}{2i} t C_{li}^l (U^i + V^i) \right) \varrho_S(e_U *_S e_V).$$

On the other hand we have again by equation (119)

$$\varrho_S(A_t (e_U *_S e_V)) = \exp \left(\frac{\lambda}{2i} t C_{li}^l (U^i + V^i) \right) \varrho_S \left(e_{\frac{1}{\lambda} H(\frac{\lambda}{i} U, \frac{\lambda}{i} V)} \right) = \exp \left(\frac{\lambda}{2i} t C_{li}^l (U^i + V^i) \right) \varrho_S(e_U *_S e_V)$$

using the shape of the Baker-Campbell-Hausdorff series and the fact that $C_{li}^l[W, X]^i = 0$ for any left-invariant vector fields W, X on G . By corollary 6.5 this implies $A_t^{-1} ((A_t e_U) *_S (A_t e_V)) = e_U *_S e_V$. Since d_S only contains derivatives with respect to vertical directions this implies that A_t is even an automorphism of $*_S$ for all $t \in \mathbb{R}$, proving the second assertion. The derivation property of d_S is an immediate consequence of (124). \square

Observe that an equation analogous to (124) does not hold for N_0 , instead we have the following lemma:

Lemma 8.4 *For any $f \in C^\infty(T^*G)$ and $\chi \in C^\infty(G)$ we have*

$$\pi^* \chi *_G f = \sum_{r=0}^{\infty} \frac{1}{r!} \left(\frac{i\lambda}{2} \right)^r \pi^* (X_{i_1} \dots X_{i_r} \chi) Z^{i_1} \dots Z^{i_r} f. \quad (125)$$

PROOF: The proof is a lengthy but straight forward computation using proposition 8.2 and the fact that all terms containing Z -derivations applied to $\pi^*\chi$ vanish due to lemma A.2. \square

Lemma 8.5 *For any left-invariant vector fields U, V on G we have*

$$\widehat{U} *_G e_V = e_V \frac{\widehat{[\frac{\lambda}{i}V, \cdot]}}{\exp([\frac{\lambda}{i}V, \cdot] - 1)} U. \quad (126)$$

Expressing the term W_r of order $(\frac{\lambda}{i})^r$ in components this reads

$$W_r(\widehat{U}, e_V) = \frac{(-1)^r}{r!} B_r P_l(Z^j \widehat{U}) C_{jk_1}^{j_1} \cdots C_{j_{r-2}k_{r-1}}^{j_{r-1}} C_{j_{r-1}k_r}^l (Z^{k_1} \cdots Z^{k_r} e_V) \quad (127)$$

where B_r denotes the r th Bernoulli number.

PROOF: By lemma 8.3 $e_U *_S e_V = e_U *_G e_V$ and therefore we have

$$\widehat{U} *_G e_V = \left. \frac{d}{dt} \right|_{t=0} e_{tU} *_G e_V = \left. \frac{d}{dt} \right|_{t=0} e_{\frac{i}{\lambda} H(t \frac{\lambda}{i} U, \frac{\lambda}{i} V)} = e_V \frac{\widehat{[\frac{\lambda}{i}V, \cdot]}}{\exp([\frac{\lambda}{i}V, \cdot] - 1)} U$$

using that for $a, b \in \mathfrak{g}$ the following is valid

$$\left. \frac{d}{dt} \right|_{t=0} H(ta, b) = \frac{\text{adb}}{\exp(\text{adb}) - 1} a.$$

Then equation (127) is a direct consequence of the Taylor series expansion of $\frac{x}{e^x - 1}$. \square

Collecting our results we get the following proposition.

Proposition 8.6 *The star product $*_G$ being related to the Fedosov star product of standard ordered type $*_S$ by means of the equivalence transformation N coincides with the product constructed in [17].*

PROOF: The assertion follows from lemma 8.4 and lemma 8.5 and the theorem proven in chapter 4 in [17]. \square

A Homogeneous connections on cotangent bundles

We now shall briefly recall some well-known basic definitions concerning horizontal and vertical lifts of vector fields, homogeneous connections and normal Darboux coordinates on cotangent bundles.

Definition A.1 *Let ∇_0 be a connection on Q then we consider the connection mapping $K : T(T^*Q) \rightarrow T^*Q$ of ∇_0 defined by*

$$K(\dot{\alpha}(0)) := \nabla_{0\partial_t} \alpha|_{t=0}$$

*for a curve α in T^*Q . It turns out that $(T\pi \times K) : T(T^*Q) \rightarrow TQ \oplus T^*Q$ is a fibrewise isomorphism. Because of this fact the notion of horizontal resp. vertical lifts with respect to ∇_0 is well-defined by: $X^h \in \Gamma(T(T^*Q))$ is called the horizontal lift of $X \in \Gamma(TQ)$ iff*

$$T\pi X^h = X \circ \pi \text{ and } K(X^h) = 0 \quad (128)$$

*resp. $\beta^v \in \Gamma(T(T^*Q))$ is called the vertical lift of $\beta \in \Gamma(T^*Q)$ iff*

$$K(\beta^v) = \beta \circ \pi \text{ and } T\pi \beta^v = 0. \quad (129)$$

Working in a local bundle Darboux chart $(q^1, \dots, q^n, p_1, \dots, p_n)$ one finds that

$$X^h = (\pi^* X^i) \partial_{q^i} + \pi^*(X^k \Gamma_{ki}^j) p_j \partial_{p_i} \quad \text{and} \quad \beta^v = (\pi^* \beta_i) \partial_{p_i}, \quad (130)$$

at which Γ_{ki}^j denotes the Christoffel symbols of ∇_0 and X^i resp. β_i the components of X resp. β . The following lemma should be well-known and is crucial throughout this paper. It can be proved easily in a local bundle chart:

Lemma A.2 *Let $T \in \Gamma(\otimes^l T^*(T^*Q))$ where $l \geq 0$ be a homogeneous tensor field of degree k , i. e. $\mathcal{L}_\xi T = kT$. Then $k \in \mathbb{Z}$ and $T = 0$ if $k < 0$ and $T = \pi^* \tilde{T}$ with $\tilde{T} \in \Gamma(\otimes^l T^*Q)$ iff $k = 0$. Moreover $[\mathcal{L}_\xi, \mathcal{L}_{\beta^v}] = -\mathcal{L}_{\beta^v}$ and $[\mathcal{L}_\xi, i_m(\beta^v)]T = -i_m(\beta^v)T$ where i_m denotes the substitution into the m th argument of T . Hence $\mathcal{L}_\xi T = kT$ implies $\mathcal{L}_\xi(\mathcal{L}_{\beta^v} T) = (k-1)T$ as well as $\mathcal{L}_\xi(i_m(\beta^v)T) = (k-1)i_m(\beta^v)T$.*

Definition A.3 *A connection ∇ on T^*Q is said to be homogeneous iff $\mathcal{L}_\xi \nabla = 0$, i. e. $\mathcal{L}_\xi \nabla_X Y - \nabla_{\mathcal{L}_\xi X} Y - \nabla_X \mathcal{L}_\xi Y = 0$ for all $X, Y \in \Gamma(T(T^*Q))$.*

Definition A.4 *The remark about $(T\pi \times K)$ being a fibrewise isomorphism justifies defining a connection on T^*Q by disposing of ∇^0 for $X, Y \in \Gamma(TQ), \beta, \gamma \in \Gamma(T^*Q)$ in the following manner:*

$$\nabla_{X^h}^0 Y^h \Big|_{\alpha_q} := (\nabla_{0X} Y)^h \Big|_{\alpha_q} + \alpha_q \left(\frac{1}{2} R(X, Y) \cdot - \frac{1}{6} (R(\cdot, X)Y + R(\cdot, Y)X) \right)^v \Big|_{\alpha_q} \quad (131)$$

$$\nabla_{X^h}^0 \beta^v \Big|_{\alpha_q} := (\nabla_{0X} \beta)^v \Big|_{\alpha_q} \quad (132)$$

$$\nabla_{\beta^v}^0 X^h \Big|_{\alpha_q} := \nabla_{\beta^v}^0 \gamma^v \Big|_{\alpha_q} := 0. \quad (133)$$

It is straight forward to check that ∇^0 is homogeneous, torsion-free and symplectic. A simple calculation using a bundle Darboux chart yields the following local expressions for the Christoffel symbols $\Gamma_{x^i x^j}^{0x^k}$ and the components of the curvature tensor $R_{x^i x^j x^l}^{0x^k} = dx^k (\nabla_{\partial_{x^j}}^0 \nabla_{\partial_{x^i}}^0 \partial_{x^l} - \nabla_{\partial_{x^l}}^0 \nabla_{\partial_{x^i}}^0 \partial_{x^j})$ of the connection ∇^0 :

$$\begin{aligned} \Gamma_{q^i q^j}^{0q^k} &= -\Gamma_{q^i p_k}^{0p_j} = -\Gamma_{p_k q^j}^{0p_i} = \pi^* \Gamma_{ij}^k \\ \Gamma_{q^i q^j}^{0p_k} &= \frac{p_a}{3} \pi^* (2\Gamma_{js}^a \Gamma_{ki}^s - \partial_{q^j} \Gamma_{ki}^a + \text{cycl.}(ijk)) \\ R_{q^k q^i q^j}^{0q^l} &= -R_{p_l q^i q^j}^{0p_k} = \pi^* R_{kij}^l \quad R_{q^k q^i p_j}^{0p_l} = \frac{1}{3} \pi^* (R_{lki}^j + R_{kli}^j) \\ R_{q^j q^k q^l}^{0p_i} &= \frac{p_a}{3} \pi^* (R_{jlk|i}^a - 3\Gamma_{is}^a R_{jlk}^s - \Gamma_{ls}^a R_{ijk}^s + \Gamma_{ks}^a R_{ijl}^s + (i \leftrightarrow j)) \end{aligned}$$

All other combinations vanish and $\dots|_i$ denotes the covariant derivative with respect to ∂_{q^i} . At this instance we want to refer to a question that was brought to our interest by M. Cahen namely whether the Ricci tensor Ric^0 corresponding to R^0 defined by $\text{Ric}^0(X, Y) := \text{tr}(Z \rightarrow R^0(Z, X)Y)$ for $X, Y \in \Gamma(T(T^*Q))$ enjoys the property that $(\nabla_X^0 \text{Ric}^0)(Y, Z) + \text{cycl.}(X, Y, Z) = 0$ for all $X, Y, Z \in \Gamma(T(T^*Q))$. It turns out by a direct but lengthy calculation that this is the fact iff the Ricci tensor Ric_0 corresponding to R_0 does so with respect to ∇_0 . In the next definition we shall briefly explain the concept of a connection on Q that is induced by a connection on T^*Q .

Definition A.5 Let $\bar{\nabla}$ be a torsion-free homogeneous connection on T^*Q . Choose any connection ∇^Q on Q and define for $X, Y \in \Gamma(TQ)$:

$$Ti(\nabla_{0X}Y) := \bar{\nabla}_{X^h}Y^h \circ i$$

where the horizontal lift is taken with respect to ∇^Q . Then $\nabla_{0X}Y$ is a well-defined vector field on Q , and does not depend on the choice of ∇^Q . We refer to ∇_0 as the connection induced by $\bar{\nabla}$.

Using this notion we can give a characterization of ∇^0 as follows:

Proposition A.6 Let ∇_0 be a torsion-free connection on Q . Among all homogeneous, torsion-free, symplectic connections $\bar{\nabla}$ on T^*Q which induce ∇_0 on Q the connection ∇^0 is uniquely characterized by the condition

$$i_a(X)\delta^{-1}\bar{R} = 0 \quad (134)$$

for all vertical vector fields X on T^*Q where $\bar{R} := \frac{1}{4}\omega_{it}\bar{R}_{jkl}^tdx^i \vee dx^j \otimes dx^k \wedge dx^l$.

PROOF: The proof mainly relies on the fact that there is a uniquely determined tensor field $B \in \Gamma(TQ \otimes \bigvee^3 T^*Q)$ such that

$$\bar{\nabla}_{X^h}Y^h|_{\alpha_q} = \nabla_{X^h}^0Y^h|_{\alpha_q} + \alpha_q(B(X, Y, _))^v|_{\alpha_q}$$

for all $X, Y \in \Gamma(TQ)$ the horizontal lifts being taken with respect to ∇_0 . Then the assertion follows by comparing \bar{R} to R^0 and the fact that (134) is fulfilled iff $B = 0$. \square

Finally we shall prove the following three lemmata:

Lemma A.7 Let (q^1, \dots, q^n) be a normal chart around q_0 with domain D^q of Q with respect to ∇_0 then for any $\alpha_q \in \pi^{-1}(D^q)$ we have

$$\Gamma_{ijk}^0(x)x^ix^jx^k\Big|_{\alpha_q} := \omega_{il}\Gamma_{jk}^{0l}(x)x^ix^jx^k\Big|_{\alpha_q} = 0,$$

where $(x^1, \dots, x^{2n}) = (q^1, \dots, q^n, p_1, \dots, p_n)$. Hence the bundle Darboux coordinates are normal Darboux coordinates with respect to ∇^0 around α_{q_0} .

PROOF: By direct calculation the assertion follows easily using the fact that (q^1, \dots, q^n) are normal coordinates on Q and the local expressions for $\Gamma_{x^ix^j}^{0x^k}$ as stated above. \square

Lemma A.8 Let M be a differentiable manifold and ∇ a connection on M then the trace of the curvature tensor R is exact i. e.

$$(\text{tr}R)(X, Y) := dx^i(R(X, Y)\partial_i) = d\alpha(X, Y) \quad \forall X, Y \in \Gamma(TM)$$

for an $\alpha \in \Gamma(T^*M)$. If $\text{tr}R = 0$ then the connection ∇ is called unimodular.

PROOF: First observe that the Levi-Civita connection ∇^{LC} of a Riemannian metric is unimodular. Now since every manifold admits such a connection we shall compare this one to ∇ . Let $S \in \Gamma(TM \otimes T^*M \otimes T^*M)$ be the uniquely determined tensor field such that $\nabla^{\text{LC}}X = \nabla_XY + S(X, Y)$. Straight forward computation using this equation and $\text{tr}R^{\text{LC}} = 0$ yields

$$(\text{tr}R)(X, Y) = -\text{tr}(Z \mapsto (S(T(X, Y), Z) + S(X, S(Y, Z)) - S(Y, S(X, Z)) + (\nabla_XS)(Y, Z) - (\nabla_YS)(X, Z)))$$

where T is the torsion of ∇ . Since $\text{tr}(Z \mapsto S(X, S(Y, Z)) - S(Y, S(X, Z))) = 0$ and since ∇ commutes with contractions we get $(\text{tr}R)(X, Y) = ((\nabla_X\alpha)(Y) - (\nabla_Y\alpha)(X)) + \alpha(T(X, Y)) = d\alpha(X, Y)$ for $\alpha(Y) := -\text{tr}(Z \mapsto S(Y, Z))$. \square

The last lemma should be well-known and is used in theorem 3.11:

Lemma A.9 *Let ∇ be a torsion-free connection on a connected manifold M . Considering normal coordinates (x^1, \dots, x^n) ($\dim(M) = n$) around an arbitrary point $q \in M$ we have the following identity for the Taylor series of a function $f \in C^\infty(M)$ with respect to ∇*

$$\tau_0(f)|_q := \sum_{r=0}^{\infty} \frac{1}{r!} D^r f \Big|_q = \sum_{r=0}^{\infty} \frac{1}{r!} \frac{\partial^r f}{\partial x^{i_1} \dots \partial x^{i_r}} \Big|_q dx^{i_1} \vee \dots \vee dx^{i_r}$$

where $D := dx^k \vee \nabla_{\partial_{x^k}}$.

PROOF: For the proof one has to observe that in any local coordinates $D^r S = (dx^i \vee \mathcal{L}_{\partial_{x^i}} - dx^i \vee dx^k \vee \Gamma_{ik}^j \partial_{x^j})^r S$ where S is an arbitrary symmetric covariant tensor field on M . By induction this is equal to $(dx^i \vee \mathcal{L}_{\partial_{x^i}})^r S + \sum_{a=1}^{N_r} C_a(K_a(S))$ where N_r is some integer, K_a are certain differential operators and $C_a = dx^{i_1} \vee \dots \vee dx^{i_{s+2}} (\partial_{x^{i_1}} \dots \partial_{x^{i_s}} \Gamma_{i_{s+1} i_{s+2}}^j) D_j$ where either $D_j = \mathcal{L}_{\partial_{x^j}}$ or $D_j = i_s(\partial_{x^j})$ and $s \geq 0$. Finally note that in normal coordinates around q we have $0 = (\partial_{x^{i_1}} \dots \partial_{x^{i_r}} \Gamma_{kl}^a)(q) dx^k \vee dx^l \vee dx^{i_1} \vee \dots \vee dx^{i_r}$ at q which is obtained by r -fold differentiation (with respect to the affine parameter) of the geodesic equation for a geodesic emanating at q . Hence C_a equals 0 at q which proves the lemma after having set $S = f$. \square

Acknowledgements

The authors would like to thank D. Arnal, M. Cahen, C. Emmrich, S. Gutt, B. Kostant, and A. Weinstein for useful discussions.

References

- [1] AGARWAL, G. S., WOLF, E.: *Calculus for Functions of Noncommuting Operators and General Phase-Space Methods in Quantum Mechanics. I. Mapping Theorems and Ordering of Functions of Noncommuting Operators*. Phys. Rev. D **2** 10 (1970), 2161-2188.
- [2] BAYEN, F., FLATO, M., FRONSDAL, C., LICHNEROWICZ, A., STERNHEIMER, D.: *Deformation Theory and Quantization*. Annals of Physics **111**, part I: 61-110, part II: 111-151 (1978).
- [3] BERTELSON, M., CAHEN, M., GUTT, S.: *Equivalence of Star Products*. Université Libre de Bruxelles, Travaux de Mathématiques, Fascicule **1**, 1–15 (1996).
- [4] BORDEMAN, M., NEUMAIER, N., WALDMANN, S.: *Homogeneous Fedosov Star Products on Cotangent Bundles II: GNS Representations, the WKB Approximation, and Applications*. Preprint Univ. Freiburg in preparation.
- [5] BORDEMAN, M., WALDMANN, S.: *A Fedosov Star Product of Wick Type for Kähler manifolds*. Preprint Univ. Freiburg FR-THEP-96/9, May 1996, and q-alg/9605012. To appear in Lett. Math. Phys.
- [6] BORDEMAN, M., WALDMANN, S.: *Formal GNS Construction and States in Deformation Quantization*. Preprint Univ. Freiburg FR-THEP-96/12, July 1996, and q-alg/9607019.
- [7] BORDEMAN, M., WALDMANN, S.: *Formal GNS Construction and WKB Expansion in Deformation Quantization*. Contribution to the Ascona Meeting on Deformation Theory, Symplectic Geometry, and Applications, June 16–22, 1996. q-alg/9611004.
- [8] CAHEN, M., GUTT, S.: *Regular \ast Representations of Lie Algebras*. Lett. Math. Phys. **6**, 395–404 (1982).

- [9] CONNES, A., FLATO, M., STERNHEIMER, D.: *Closed Star Products and Cyclic Cohomology* Lett. Math. Phys. **24**, 1-12 (1992).
- [10] DEWILDE, M., LECOMTE, P. B. A.: *Star Products on Cotangent Bundles*. Lett. Math. Phys. **7**, 235–241 (1983).
- [11] DEWILDE, M., LECOMTE, P. B. A.: *Existence of star products and of formal deformations of the Poisson Lie Algebra of arbitrary symplectic manifolds*. Lett. Math. Phys. **7**, 487-496 (1983).
- [12] DEWILDE, M., LECOMTE, P. B. A.: *Formal Deformations of the Poisson Lie Algebra of a Symplectic Manifold and Star Products. Existence, Equivalence, Derivations*. in: HAZEWINKEL, M., GERSTENHABER, M. (EDS): *Deformation Theory of Algebras and Structures and Applications*. Kluwer, Dordrecht 1988.
- [13] DRINFEL'D, V. G.: *On constant quasiclassical solutions of the Yang-Baxter quantum equation*. Soviet Math. Dokl. **28**, 667–671 (1983).
- [14] EMMRICH, C., WEINSTEIN, A.: *The differential geometry of Fedosov's quantization*. Preprint Univ. California, Berkley, Nov. 1993, and hep-th/9311094.
- [15] FEDOSOV, B.: *A Simple Geometrical Construction of Deformation Quantization*. J. Diff. Geom. **40**, 213-238 (1994).
- [16] FEDOSOV, B.: *Deformation Quantization and Index Theory*. Akademie Verlag, Berlin 1996.
- [17] GUTT, S.: *An explicit \ast -Product on the cotangent bundle of a Lie Group*. Lett. Math. Phys. **7**, 249–258 (1983).
- [18] NEST, R., TSYGAN, B.: *Algebraic Index Theorem*. Commun. Math. Phys. **172**, 223–262 (1995).
- [19] NEST, R., TSYGAN, B.: *Algebraic Index Theorem for Families*. Adv. Math. **113**, 151–205 (1995).
- [20] PFLAUM, M. J.: *Local Analysis of Deformation Quantization*. Ph.D. thesis, Fakultät für Mathematik der Ludwig-Maximilians-Universität, München, 1995.
- [21] PFLAUM, M. J.: *The Normal Symbol on Riemannian Manifolds*. Preprint dg-ga/9612011.
- [22] UNDERHILL, J.: *Quantization on a manifold with connection*. J. Math. Phys. **19**, 1932–1935 (1978).
- [23] WIDOM, H.: *Families of Pseudodifferential Operators*. in: GOHBERG, I., KAC, M. (EDS): *Topics in Functional Analysis*. Academic Press, New York 1978.
- [24] WIDOM, H.: *A Complete Symbol Calculus for Pseudodifferential Operators*. Bull. Sci. Math. **104**, 19–63 (1980).
- [25] WOODHOUSE, N.: *Geometric Quantization*. Clarendon Press, Oxford 1980.